An airline revenue management pricing game with seat allocation

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Abstract: This paper studies a horizontal fare-pricing competition between two airlines having a single flight leg. Two distinct scenarios are considered. First, the two airlines price competition for the pre-committed booking limits is analysed. The problem is studied under deterministic price sensitive demands. The existence of unique pricing strategies at Nash equilibrium is shown. In the second scenario, a joint seat allocation and fare-pricing competition model for stochastic demand is proposed. A numerical analysis is presented to demonstrate the impacts of various market conditions on the payoffs, booking limits and pricing strategies of the competing airlines.

Keywords: airline revenue management; game theory; Nash equilibrium; pricing; seat inventory control.

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1 Introduction

Revenue Management (RM), also called Yield Management (YM), is a practice of managing perishable assets by controlling their availabilities and/or prices with an objective to maximise the total revenue. RM addresses two important issues in demand management: determination of

1 price

2 quantity.

A pricing decision deals with product's list price, auction price and markdown price (discount). A quantity decision deals with the allocation of capacity or output into different segments, product or classes. It also includes the decision whether to accept or reject a customer demand. Based on these two important issues, the RM practice is classified into two major units:

1 quantity-based RM

2 price-based RM.

While the quantity-based RM is mostly practised in airline industry, the price-based RM is frequently used in retail industry. In quantity-based RM, the objective is to optimally allocate available capacity into different demand classes.

Airline RM studies mainly focus on the optimal determination of the capacity allocated for each fare class. Each fare class represents a different discount level and service offerings. In a given flight, the cabin capacity is allocated among available fare classes using booking limits. Booking limits are either partitioned or nested. In the partitioned booking limits, the capacity is completely segregated from other fare classes. On the other hand, in the nested booking-limits, the capacity is also available for higher fare classes in a hierarchical order, i.e. low-fare capacity is always available to high-fare customers. Nested limits help in avoiding the problem of cabin capacity being unavailable to a high-fare customer while it is available to a low-fare customer.

A single-leg horizontal competition is a competition among two or several airlines in a single non-stop flight with the same origin and destination and similar flight times. This paper considers a single-leg horizontal competition using joint control approach on seat allocation and fare pricing. The approach presented here is novel as it is able to control the booking limits and fare pricing jointly while considering the market competition. The fact that the capacity is a function of the price and the availability is a function of demand has not been addressed in the literature in detail. Furthermore, not many works address the pricing and booking limit determination within the competitive market conditions. Hence, the work presented in this paper is significant, since it aims at addressing the problem of joint determination of pricing strategies and booking limits in a duopoly market.

The remainder of this paper is organised as follows. A brief summary on relevant literature is discussed in Section 2. In Section 3, a pricing competition among airlines in a duopoly environment is discussed. In Section 4, effects of stochastic demand are studied and two modelling approaches for jointly determining nested booking limits and fare prices are developed. In Section 5, the determination of pricing strategies and booking limits at (near) Nash equilibrium for competing airlines is shown in a numerical study. Finally, conclusions and future research directions are outlined in Section 6.

2 Related literature

The problem of RM has attracted several scholars as well as large airline companies for a period of 30 years. The research in airline RM is classified into four branches:

1 demand forecasting

2 overbooking

3 seat inventory control

4 pricing.

These issues are distinct but closely related. The forecasting, overbooking and seat inventory control have received more attention than pricing in the literature. Today airline RM practice includes decision support tools to help overbooking, seat allocation and forecasting (Cote, Marcotte and Savard, 2003). However, the coverage area of RM works is not limited to airline and retail industry. Industries such as internet providers and broadcasting companies are also beneficiaries of the RM studies (Mangàni, 2007; Kimms and Müller-Bungart, 2007). A comprehensive overview of literature related to airline RM can be found in McGill and Ryzin (1999). The literature review presented in this paper covers selected research on seat allocation, fare pricing, works in close resemblance with joint control of seat allocation with fare pricing and competition analysis in airline RM.

Some early work in seat inventory control research is due to Littlewood (1972) with an application to the airline industry. Belobaba (1987, 1989) extended Littlewood's (1972) work and proposed the commercially most practised Expected Marginal Seat Revenue (EMSR) heuristic. Brumelle and Walczak (2003) studied the dynamic nature of the RM problem with multiple demands. Bertsimas and Popescu (2003) studied the seat allocation problem in a flight network and proposed an approximate dynamic programming approach for network RM.

Pricing strategies are considered as efficient tools for market competition. Literature existing in economics provides a substantial study on pricing. In general, the literature in pricing is divided into two segments: static pricing and dynamic pricing. In this work, we only discuss the static price competition. Bertrand–Edgeworth is a primary model in price competition. It allocates deterministic demand among competing firms with a fixed capacity based on their prices. Some closely related papers extending the work of Bertrand–Edgeworth are Kreps and Scheinkman (1983), Allen and Hellwig (1986), and Deneckere and Kovenock (1996). An overview of pricing research in the context of RM is done by Bitran and Caldentey (2003).

There are some published works considering economic interaction in the aviation market, strategic airline market entry decisions and airline schedule design. Hotelling (1929) studied the problem of an airline's scheduling decision under competition using variant of spatial model. Some other empirical works are Borenstein and Netz (1999) and Richard (2003). This group also includes an interactive airline scheduling model of Dobson and Lederer (1994) and network design model of Lederer and Mambimadom (1998). Gottinger (2007) explores the competitive positioning of organisations through network competition on the basis of alliance formation. These papers mainly consider competition in aviation market at large scale, but ignore any seat allocation decision or integrated framework for fare pricing and seat allocation under competitive market conditions.

An effort to determine fare pricing and seat allocation jointly in airline RM is due to Weatherford (1997). Their model ignores the market competition. Li and Oum (1998) describe a seat allocation game for fixed fare prices. In their work, a symmetric equilibrium for the game is identified. An extended work on the seat allocation game with fixed fare pricing in a duopoly market is presented in Netessine and Shumsky (2005). The authors studied the seat allocation game under horizontal and vertical competition (different airlines fly on different legs in a multiple-leg itinerary). Customers are allowed to substitute among competing airlines within the same fare class. A revenue sharing policy is also described in the case of vertical competition. Chen, Yan and Yao (2004) studied the competitive Newsvendor problem with joint strategy of seat allocation and pricing for a single commodity. More recent work that considers multiple firms competing for customers in the context of RM with a single commodity is due to Dai et al. (2005). A sensitivity analysis of pricing strategies at Nash equilibrium is presented under various deterministic and stochastic price-sensitive demand conditions. Finally, the pricing problem in the content of Newsvendor problem under probabilistic demand conditions is studied in Petruzzi and Dada (1999). They integrated the probabilistic factor into deterministic demand using additive and multiplicative modelling approaches.

The competitive fare pricing and seat allocation studies in the airline RM problem are very recent research topics and there is a growing interest in this area. The study presented in this paper is distinguished from Netessine and Shumsky's (2005) work by its ability to determine both the seat allocation and fare pricing under market competition.

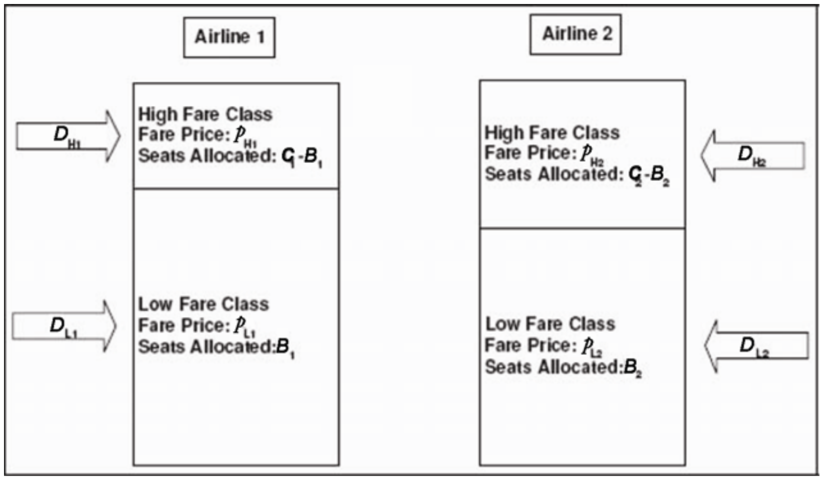
3 Price competition in Revenue Management

The problem is illustrated with a single flight leg competition between two airlines in a duopoly market where two airlines are offering two fare classes to customers. The market is also divided into only two customer classes and segmentation is considered perfect, i.e. a high-fare customer does not request a low-fare ticket and vice versa. Customer diversion is also not considered in this model. Customers are considered rational and the fare price is the only factor to calibrate the rationality of the customers in each fare class. We first consider a scenario when two airlines arbitrarily pre-commit the seat allocation (booking limits) for each fare class for a known flight capacity. Once the booking limits are pre-committed, both airlines compete for customers in each fare class in a non-cooperative duopoly environment using their pricing strategies only. The problem is modelled using game theoretic approach. Other assumptions used in the model are:

1. both airlines offer single non-stop flight and they do not cooperate for a joint revenue maximisation
2. flight capacities offered by two airlines are known and fixed
3. customer demand is observed sequentially, i.e. the low-fare class demand is observed before the high-fare class demand.

A duopoly fare pricing with allocation competition model is presented in Figure 1.

Figure 1 Two airlines competition for two fare classes



Let us now define the notations that are used in the modelling of this problem:

P_{ci} Fare price offered in the fare class $c = \{L, H\}$ by airline $i = 1, 2$

C_i Total flight capacity of airline $i = 1, 2$

B_i Booking limit for the low-fare price committed by airline $i = 1, 2$

$\prod_i$ Total revenue generated by airline $i = 1, 2$.

Let D_{ci} be the riskless demand observed by airline i when its fare is P_{ci} and its competitor's fare is P_{cj} for the booking class c, $\forall i = \{1, 2\}, j = \{1, 2\}$. D_{ci} is a continuous and twice differentiable function. It is bounded in $P_{ci} \in [\underline{P}_{ci}, \overline{P}_{ci}]$ and $P_{cj} \in [\underline{P}_{cj}, \overline{P}_{cj}]$. Also $B_i \in [0, C_i]$, $\forall i = \{1, 2\}$. Also, it is not common to assume D_{ci} is a supermodular function of fare prices (Topkis, 1978), which is also observed here. Hence, the riskless demand is given in a function form as: $D_{ci} = D_{ci}(P_{ci}, P_{cj})$ $\forall i = \{1, 2\}, j = \{1, 2\}$. In the following sections, the formulation of price competition problem and its solution methodology are discussed in depth.

Once the booking limits are known, airlines compete in each fare class using their fare pricing strategies. For a pre-determined booking limit, the fare-pricing competition model has the following revenue function.

$$\Pi_i = P_{Li} \min\{B_i, D_{Li}\} + P_{Hi} \min\{D_{Hi}, C_i - B_i\} \tag{1}$$

where P_{Li} and P_{Hi} are prices, and D_{Li} and D_{Hi} are riskless demands, respectively.

An alternative form for the above revenue function is

$$\Pi_i = P_{Li}B_i - P_{Li}[B_i - D_{Li}]^+ + P_{Hi}(C_i - B_i) - P_{Hi}[C_i - B_i - D_{Hi}]^+ \tag{2}$$

where $[x]^+ = \max\{0, x\}$, $\forall x \in R$.

For each fare class, the following assumptions for the demand functions are made:

1 *Assumption 1:* $\frac{\partial D_{Li}}{\partial P_{Li}} < 0, \frac{\partial D_{Hi}}{\partial P_{Hi}} < 0, \quad \forall\, i = \{1, 2\}$.

2 *Assumption 2:* $\frac{\partial D_{Li}}{\partial P_{Lj}} > 0, \frac{\partial D_{Hi}}{\partial P_{Hj}} > 0, \quad \forall\, i, j = \{1, 2\}, i \neq j$.

3 *Assumption 3:* $-D_{Li}$ and $-D_{Hi}$ are sub modular in (P_{L1}, P_{L2}) and (P_{H1}, P_{H2}) $i = \{1, 2\}$, respectively.

In Assumption 1, it is stated that the demand has increasing price elasticity, i.e. demand decreases with an increase in price. In Assumption 2, it is stated that the demand for a fare class increases with the increase in the fare pricing of its rival airline. Assumption 3 implies that the low-fare class demand increases when the competing airline's fare P_{L2} is decreased, hence, $[D_{L1}(P_{L1}, P_{L2}^l) - D_{L1}(P_{L1}, P_{L2}^h)] < 0.$ Indexes l and h refers to low and high limits, respectively. These assumptions are first stated by Topkis (1979) and are commonly observed in price competition research of substitutable services/product. Some other related works using the similar assumptions are Bernstein and Federgruen (2004a,b, 2005) and also in Dai et al. (2005). Topkis (1979) also uses the following results to prove the existence of Nash equilibrium, which is a measure we also seek for the competition model discussed in this paper.

Definition 1: A function $f(x_1, x_2)$ is submodular in (x_1, x_2), if $f(x_1^l, x_2) - f(x_1^h, x_2)$ is non-decreasing in x_2 for all $x_1^l \leq x_1^h$. A function $f(x_1, x_2)$ is supermodular if $-f(x_1, x_2)$ is submodular, where l is low and h is high.

Lemma 1: Suppose $f(x_1, x_2)$ is twice differentiable, then $f(x_1, x_2)$ is submodular in (x_1, x_2), if and only if $\frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2} \leq 0.$

Lemma 2: A function $f(x_1, x_2)$ is supermodular (submodular) in (x_1, x_2) if and only if it is submodular (supermodular) in $f(x_1, -x_2)$.

Consequently, the revenue of an airline can further be analysed using the variable pricing transformation as suggested in Lippman and McCardle (1997) and the results stated in Lemma 2. Assuming $Z_{Li} = -P_{Li}$ $\forall i = \{1, 2\}$ and $Z_{Hi} = -P_{Hi}$ $\forall i = \{1, 2\}$, we restate the previous assumptions in the following forms:

1 *Assumption 1:* $\frac{\partial D_{Li}}{\partial P_{Li}} < 0, \frac{\partial D_{Hi}}{\partial P_{Hi}} < 0, \quad \forall\, i = \{1, 2\}$.

2 *Assumption 2:* $\frac{\partial D_{Li}}{\partial Z_{Lj}} < 0, \frac{\partial D_{Hi}}{\partial Z_{Hj}} < 0, \quad \forall\, i, j = \{1, 2\}, i \neq j$.

3 *Assumption 3:* $-D_{L\,i}(P_{Li}, Z_{Li})$ and $-D_{H\,i}(P_{Hi}, Z_{Hi})$ are supermodular in (P_{L1}, Z_{L2}) and (P_{H1}, Z_{H2}) $i = \{1, 2\}$, respectively.

For the airline i with the pre-committed booking limits, Lemma 3, as suggested in Topkis (1978), enables us to identify the supermodularity of the competing airlines' revenue functions.

Lemma 3: Suppose $g(x_1, x_2)$ is monotonic in both x_1 and x_2 and is a supermodular function in (x_1, x_2). Furthermore, $f(.)$ is an increasing convex function. Then $f(g(x_1, x_2))$ is a supermodular function in (x_1, x_2).

In Proposition 1, we show that the total revenue of an airline in competition under pre-committed booking limit is supermodular. Later for the same problem, we show that the Nash equilibrium is also unique.

Proposition 1: In the two airlines' pricing game when the demand is considered deterministic, there exists a unique Nash equilibrium if $\left|\frac{\partial^2 \Pi_{Li}}{\partial P_{Li} \partial P_{Lj}}\right| < \left|\frac{\partial^2 \Pi_{Li}}{\partial P_{Li}{}^2}\right|$ and $\left|\frac{\partial^2 \Pi_{Hi}}{\partial P_{Hi} \partial P_{Hj}}\right| < \left|\frac{\partial^2 \Pi_{Hi}}{\partial P_{Hi}{}^2}\right|$.

From a theorem given in Topkis (1979), we know that if the strategy space is a complete lattice, the joint payoff function is upper-semicontinuous, and each player's payoff is supermodular. Therefore, each player's best response is increasing in the other player's strategy. This can be explained as a strategy, which results an increase in the payoff of one player also resulting a gain in the payoff of the other player. When the best response exhibits this monotonicity property, the players' strategies are said to be strategic complements, and the existence of Nash equilibrium is easy to establish (see Lippman and McCardle, 1997). Assuming the demand is deterministic, and then the Proposition 1 is sufficient to show the existence of a unique Nash equilibrium. Moulin (1986) further suggests that the above condition is sufficient for the uniqueness of Nash equilibrium. The Proposition 1 gives the slopes of players' best responses and the slops never exceed one in the absolute value.

In the literature, linear and logit (see Chen et al., 2004) are the most commonly used techniques to model price-sensitive demand. In this work, we also propose a linear model to represent price-sensitive demand. We define linear functions to model deterministic and price sensitive demands for each fare class as follows:

$$\begin{aligned} &D_{ci} = \alpha_{ci} - \beta_{ci} P_{ci} + \theta_{cij} P_{cj}, \quad \forall\ \beta_{ci} > \theta_{cij} \\ &\text{and } \beta_{ci}, \theta_{cij} \geq 0; i \neq j; i, j = \{1, 2\}; c = \{L, H\} \end{aligned} \tag{3}$$

where c is the booking-class, α_{ci} is the average price-insensitive demand for the airline i and β_{ci} is the mean impact of price variation on demand for the airline i with a unit change in the price. The mean impact on airline $-i$'s demand due to a unit variation in the price of its rival airline j is given by θ_{cij}. Here, it is also assumed that $\beta_{ci} > \theta_{cij}$ for $i \neq j$; $i, j = \{1, 2\}$; $c = \{L, H\}$; otherwise an airline can increase demands while still increasing the fare price. The above argument is correct for both low- and high-fare booking classes.

It is easy to verify that the condition stated in Proposition 1 is always true for previously mentioned linear demand function in each fare class; hence there exist a unique Nash equilibrium. As mentioned earlier, the total revenue function is decomposed into two separate revenue functions:

1 the revenue generated from low-fare class $\prod_{Li}$

2 revenue generated from high-fare class $\prod_{Hi}$ $\forall i = \{1, 2\}$.

In Section 3.1, a detailed sensitivity analysis for the low-fare class is discussed. Via an analogy with the low-fare class analysis, conclusions for the high-fare class are also derived. Finally, the section is concluded with discussion on the multi-fare class extension of the model.

3.1 Low-fare pricing competition

Since the booking limits are pre-committed, (B_i) is assumed to be known. Hence, the low-fare revenue function for airline-1 is:

$$\Pi_{L1} = \begin{cases} P_{L1}D_{L1}, & \text{when } D_{L1} < B_1 \\ P_{L1}B_1, & \text{otherwise.} \end{cases} \tag{4}$$

From Equation (4), we establish two distinct response functions for airline-1. When the demand is $D_{L1} < B_1$ airline's revenue is:

$$\Pi_{L1} = P_{L1}\left(\alpha_{L1} - P_{L1}\,\beta_{L1} + P_{L2}\,\theta_{L12}\right) \tag{5}$$

where $\prod_{L1}$ is concave for a given P_{L2}. Now, we can determine the best response function of airline-1 under the defined situation by applying the first order optimality condition as:

$$\alpha_{L1} - 2\,P_{L1}\,\beta_{L1} + P_{L2}\,\theta_{L12} = 0. \tag{6}$$

In graphical representation, plane (P_{L1}, P_{L2}), the line presented in Equation (6) has slope $\frac{2\beta_{L1}}{\theta_{L12}} > 2$, and it passes through point ($\alpha_{L1}/2\beta_{L1}$, 0). This is the best response function of airline-1 when its low-fare demand is less than its booking limit, B_1. We call this as 'the Low-fare class Capacity Greater than the Demand for airline-1' (LCGD1).

A contrary case to LCGD1 is 'the Low-fare class Capacity Less than the Demand for airline-1' (LCLD1), i.e. $D_{L1} \geq B_1$. The revenue function under this condition is $\Pi_{L1} = P_{L1}B_1$. The best response function becomes $B_1 = \alpha_{L1} - \beta_{L1}P_{L1} + \theta_{L12}P_{L2}$. Let P'_{L1} is the price such that $D_{L1} = B_1$, then the price is:

$$P'_{L1} = \frac{\alpha_{L1} - B_1 + P_{L2}\theta_{L12}}{\beta_{L1}} \tag{7}$$

The slope of the best response function is $\beta_{L1}/\theta_{L1} > 1$, which passes through $((\alpha_{L1} - B_1)/\beta_{L1}, 0)$. Based on P'_{L1}, we identify LCGD1 and LCLD1 situations when the low-fare pricing offered by airline-1 is either $P_{L1} > P'_{L1}$ or $P_{L1} \leq P'_{L1}$, respectively. Depending upon the booking limit B_1, the LCGD1 and LCLD1 situations can also be identified. When $((-B_1 + \alpha_{L1})/\beta_{L1}) \leq (\alpha_{L1}/2\beta_{L1})$, i.e. $B_1 \geq (\alpha_{L1}/2)$, then airline-1's best response includes both LCGD1 and LCLD1. However, for $B_1 < (\alpha_{L1}/2)$, the best response function of airline-1 contains only LCLD1.

A similar analysis can be done for the airline-2 (competitor). When airline-2 pre-commits a booking limit B_2, its low-fare revenue function is:

$$\Pi_{L2} = \begin{cases} P_{L2}D_{L2}, & \text{when } D_{L2} < B_2 \\ P_{L2}B_2, & \text{otherwise} \end{cases} \tag{8}$$

Likewise, airline-1 and airline-2 also have two situations in the low-fare class demand. For the situation when 'the Low-fare class Capacity Greater than the Demand for airline-2' (LCGD2) the best response function would be $\alpha_{L2} - 2P_{L2}\beta_{L2} + P_{L1}\theta_{L21} = 0$. For the case of 'the Low-fare class Capacity Less than the Demand for airline-2' (LCLD2), the best response function would be $B_2 = \alpha_{L2}\ \beta_{L2}P_{L2} + \theta_{L21}P_{L1}$. In following, we present various cases that airlines may experience in this deterministic low-fare pricing competition. The pricing strategies at Nash equilibrium for all the cases are given in the Table 1.

Case 1: $D_{L1} > B_1$ and $D_{L2} > B_2$

In this case, both airlines experience a demand greater than their capacities. Thus, the case is LCLD1 and LCLD2.

Case 2: $D_{L1} \le B_1$ and $D_{L2} > B_2$

Table 1 Low-fare pricing strategies at varying market conditions

	$D_{L1} > B_1$ *and LCLD1*	$D_{L1} \le B_1$ *and (LCLD1 or LCGD1)*
$D_{L2} >$ B$_2$ and LCLD2	$P_{L1} = \frac{(-B_1+\alpha_{L1})\beta_{L2} + (-B_2+\alpha_{L2})\theta_{L12}}{\beta_{L1}\beta_{L2} - \theta_{L12}\theta_{21}}$ $P_{L2} = \frac{(-B_2+\alpha_{L2})\beta_{L1} + (-B_1+\alpha_{L1})\theta_{L21}}{\beta_{L1}\beta_{L2} - \theta_{L12}\theta_{21}}$	$P_{L1} = \frac{\alpha_{L1}\beta_{L2} + (-B_2+\alpha_{L2})\theta_{L12}}{2\beta_{L1}\beta_{L2} - \beta_{L1}\beta_{L2} - \theta_{L12}\theta_{21}}$ $P_{L2} = \frac{2(-B_2+\alpha_{L2})\beta_{L1} + \alpha_{L1}\theta_{L21}}{2\beta_{L1}\beta_{L2} - \theta_{L12}\theta_{21}}$
$D_{L2} \le$ B$_2$ and (LCLD2 or LCGD2)	$P_{L1} = \frac{2(-B_1+\alpha_{L1})\beta_{L2} + \alpha_{L2}\theta_{L12}}{2\beta_{L1}\beta_{L2} - \theta_{L12}\theta_{21}}$ $P_{L2} = \frac{\alpha_{L2}\beta_{L1} + (-B_1+\alpha_{L1})\theta_{L21}}{2\beta_{L1}\beta_{L2} - \theta_{L12}\theta_{21}}$	If $B_1 \ge B_1'$ and $B_2 \ge B_2'$ (Case 4.4) $P_{L1} = \frac{2\alpha_{L1}\beta_{L2} + \alpha_{L2}\theta_{L12}}{4\beta_{L1}\beta_{L2} - \theta_{L12}\theta_{L21}}$ $P_{L2} = \frac{2\alpha_{L2}\beta_{L1} + \alpha_{L1}\theta_{L21}}{4\beta_{L1}\beta_{L2} - \theta_{L12}\theta_{L21}}$ For other conditions, see the Cases 4.1–4.3

Airline-2 is still facing LCLD2 but airline-1 is experiencing two responses, LCLD1 and LCGD1. Pricing strategies, given in the Table 1, are true with the following condition:

$$\begin{aligned} &B_1(2\beta_{L1}\beta_{L2}^2 + 2\beta_{L1}\beta_{L2}\theta_{L21} - \beta_{L2}\theta_{L12}\theta_{L21} - \theta_{L12}\theta_{L21}^2) \\ &+B_2(\beta_{L1}\beta_{L2}\theta_{L12} + \beta_{L1}\theta_{L12}\theta_{L21}) \ge \lambda_1 \end{aligned} \tag{9}$$

where

$$\lambda_1 = \alpha_{L1}\beta_{L1}\beta_{L2}^2 - \alpha_{L2}\beta_{L1}\beta_{L2}\theta_{L12} + \alpha_{L1}\beta_{L1}\beta_{L2}\theta_{L21} + \alpha_{L2}\beta_{L1}\beta_{L12}\theta_{L21}. \tag{10}$$

Case 3: $D_{L1} > B_1$ and $D_{L2} \le B_2$

The situation is contrary to Case 2. While airline-2 is experiencing both LCLD2 and LCGD2, airline-1 is experiencing only LCLD2. Pricing strategies, given in the Table 1, are true with the following condition:

$$B_1(-(\beta_{L1}\beta_{L2}\theta_{L21}) - \beta_{L2}\theta_{12}\theta_{L21}) + B_2(-2\beta_{L1}^2\beta_{L2} - 2\beta_{L1}\beta_{L2}\theta_{L12} + \beta_{L1}\theta_{L12}\theta_{L21} + \theta_{L12}^2\theta_{L21}) \ge \lambda_2 \tag{11}$$

where

$$\lambda_2 = -\alpha_{L2}\beta_{L1}^2\beta_{L2} - \alpha_{L2}\beta_{L1}\beta_{L2}\theta_{L12} - \alpha_{L1}\beta_{L1}\beta_{L2}\theta_{L21} - \alpha_{L1}\beta_{L2}\theta_{L12}\theta_{L21}. \tag{12}$$

Case 4. $D_{L1} \le B_1$ and $D_{L2} \le B_2$

In this case, each airline faces two situations. Airline-1 faces both LCLD1 and LCGD1 situations. The intersection of these two distinct responses is $\left(B_1/\beta_{L1}, (2B_1 - \alpha_{L1})/\theta_{L12}\right)$. We determine a condition such that the best response function of airline-2 when facing LCGD2 passes through this intersection. This condition is achieved by modifying the pre-committed booking limit B_1 to a new value B_1'. The modified value is:

$$B_1' = \frac{\beta_{L1}(2\alpha_{L1}\beta_{L2} + \alpha_{L2}\theta_{L12})}{4\beta_{L1}\beta_{L2} - \theta_{L12}\theta_{L21}}. \tag{13}$$

A similar analysis is done for airline-2 when it is experiencing both LCLD2 and LCGD2. The intersection of the best response functions of airline facing LCLD2 and LCGD2 is $((2B_2 - \alpha_{L2})/\theta_{L21},\ B_2/\beta_{L2})$. We determine a modified value of B_2' such that the best response function of airline-1 passes through aforementioned intersection point when it is facing LCGD1. The modified value of B_2' in this situation is

$$B_2' = \frac{\beta_{L2}(2\alpha_{L2}\beta_{L1} + \alpha_{L1}\theta_{L21})}{4\beta_{L1}\beta_{L2} - \theta_{L12}\theta_{L21}}. \tag{14}$$

It is easy to verify that $B_1' > \frac{\alpha_{L1}}{2}$ and similarly $B_2' > \frac{\alpha_{L2}}{2}$. Depending upon B_1 and B_2 there can be four sub-cases:

Case 4.1: $\frac{\alpha_{L1}}{2} \le B_1 < B_1'$ and $\frac{\alpha_{L2}}{2} \le B_2 < B_2'$

In this case, airline-1 and -2 face LCLD1 and LCLD2, respectively, which is aforementioned Case 1.

Case 4.2: $B_1 \ge B_1'$ and $\frac{\alpha_{L2}}{2} \le B_2 < B_2'$

There are two possibilities. Similar to Case 2, we know that if the Equation (9) holds (where λ_1 is driven in Equation (10)), then the pricing strategies at Nash equilibrium are identified in Case 2. Otherwise, airlines will face LCLD1 and LCLD2, so the pricing at Nash equilibrium is same as in Case 1.

Case 4.3: $\frac{\alpha_{L1}}{2} \leq B_1 < B_1'$ and $B_2 \geq B_2'$

Again there are two possibilities. Similar to the Case 3, we know that if Equation (11) for the given λ_2 holds, then the pricing strategies at Nash equilibrium are as identified in Case 3. Otherwise, airlines observe LCLD1 and LCLD2 which is the Case 1.

Case 4.4: $B_1 \geq B_1'$ and $B_2 \geq B_2'$

In this case, airlines experience LCGD1 and LCGD2. Hence, the pricing strategies at Nash equilibrium are found by solving the best response functions of the two airlines as shown in the Table 1:

3.2 High-fare pricing competition

Under the pre-committed booking limit assumption, the allocated capacity for the high-fare class of airline *i* is $C_i - B_i \ \forall i = \{1, 2\}$. To analyse the high-fare class competition, we consider that the airline-1 is observing two distinct customer demand behaviours. The first is 'High-fare Capacity is Less than the Demand for airline-1' (HCLD1) and the second one is 'High-fare Capacity is Greater than the Demand for airline-1' (HCGD1). In the case of HCLD1, the best response function of airline-1 is $C_1 - D_{H1} = \alpha_{H1} - P_{H1}\beta_{H1} + P_{H2}\theta_{H12}$. However in the case of HCGD1, the best response function becomes $\alpha_{H1} - 2P_{H1}\beta_{H1} + P_{H2}\theta_{H12} = 0$. For airline-2, the best response functions facing HCLD2 and HCGD2 are $C_2 - D_{H2} = \alpha_{H2} - P_{H2}\beta_{H2} + P_{H1}\theta_{H21}$ and $\alpha_{H2} - 2P_{H2}\beta_{H2} + P_{H1}\theta_{H21} = 0$, respectively. Likewise in the low-fare pricing analysis, the response functions for high-fare class price can also be used to derive competitive high-fare prices. The summary of pricing strategies for the possible demand conditions are given in Table 2.

Table 2 High-fare pricing strategies at varying market conditions

	$D_{H1} > (C_1 - B_1)$ and HCLD1	$D_{H1} \leq (C_1 - B_1)$ and (HCLD1 or HCGD1)
$D_{H2} >$ $(C_2 - B_2)$ and HCLD2	$P_{H1} = \frac{(B_1 - C_1 + \alpha_{H1})\beta_{H2} + (B_2 - C_2 + \alpha_{H2})\theta_{H12}}{\beta_{H1}\beta_{H2} - \theta_{H12}\theta_{H21}}$ $P_{H2} = \frac{(B_2 + \alpha_{L2})\beta_{L1} + (-B_1 + \alpha_{L1})\theta_{L21}}{\beta_{H1}\beta_{H2} - \theta_{H12}\theta_{H21}}$	$P_{H1} = \frac{\alpha_{H1}\beta_{H2} + (B_2 - C_2 + \alpha_{H2})\theta_{H12}}{2\beta_{H1}\beta_{H2} - \theta_{H12}\theta_{H21}}$ $P_{H2} = \frac{2(B_2 - C_2 + \alpha_{H2})\beta_{H1} + \alpha_{H1}\theta_{H21}}{\beta_{H1}\beta_{H2} - \theta_{H12}\theta_{H21}}$
$D_{H2} \leq$ $(C_2 - B_2)$ and (HCLD2 or HCGD2)	$P_{H1} = \frac{2(B_1 - C_1 + \alpha_{H1})\beta_{H2} + \alpha_{H2}\theta_{H12}}{2\beta_{H1}\beta_{H2} - \theta_{H12}\theta_{H21}}$ $P_{H2} = \frac{\alpha_{H2}\beta_{H1} + (B_1 - C_1 + \alpha_{H1})\theta_{H21}}{2\beta_{H1}\beta_{H2} - \theta_{H12}\theta_{H21}}$	If $B_1 \geq B_1'$ and $B_2 \geq B_2'$ (Case 4.4) $P_{H1} = \frac{2\alpha_{H1}\beta_{H2} + \alpha_{H2}\theta_{H12}}{4\beta_{H1}\beta_{H2} - \theta_{H12}\theta_{H21}}$ $P_{H2} = \frac{2\alpha_{H2}\beta_{H1} + \alpha_{H1}\theta_{H21}}{4\beta_{H1}\beta_{H2} - \theta_{H12}\theta_{H21}}$

3.3 Extension to multi-fare class problem

Total expected revenue of multi-fare class problem is:

$$\Pi_i = \sum_{i=1}^{n} (P_{ci} \min\{B_{ci}, D_{ci}\}). \tag{15}$$

Further, the above revenue function can be partitioned for any particular booking class (booking class *m* in our case):

$$\Pi_i = \sum_{c=1}^{m-1} (P_{ci} \min\{B_{ci}, D_{ci}\}) + P_{mi} \min\{B_{mi}, D_{mi}\} + \sum_{c=m+1}^{n} (P_{ci} \min\{B_{ci}, D_{ci}\}) \quad (16)$$

where n is the number of booking classes and m is the booking class being analysed. Since, the cabin capacity is divided among the fare classes and the booking limits are known, the booking class m can be analysed similar to both low- and high-fare class analysis discussed earlier in the text.

Up to this point, the RM problem in competition in a duopoly market is analysed using a price-based approach where the quantity is pre-determined. However, in practice, the observed demand, which is probabilistic, is the dominant factor for determining both price and capacity. Furthermore, the demand of a commodity can be given as a function of price and competition. Hence, the real question for RM to tackle is the joint control of both price and capacity for a price-sensitive demand. Therefore, in Section 4, we extend the RM model, discussed in the Section 3, to consider an integrated framework towards price and quantity-based RM in the airlines industry. Accordingly, in Section 4, we model the RM problem for a probabilistic demand. Because the developed stochastic RM model does not have a closed form, we have simplified the model using an additive and multiplicative modelling approach as suggested in the literature and solve them numerically.

4 Fare-pricing model with seat allocation under pricing competition

In this section, we attempt to find the optimal seat allocation between two airlines having a horizontal fare-pricing competition. Airlines jointly make decisions on seat allocations (booking limits) and fare pricing. Let S_{ci} be the stochastic demands to airline i for its fare class c. There are two fare classes, Low (L) and High (H), thus $c = \{L, H\}$. More precisely, S_{ci} is $S_{ci}(D_{ci}, \xi_{ci})$, which is the function of riskless (deterministic) demand in fare class c, D_{ci}, and a stochastic demand factor ξ_{ci}, $c = \{L, H\}$. An essential assumption to model the fare-pricing game jointly with seat allocation is that the random variable ξ_{ci} is independent of fare prices and follows a continuous probability distribution function ϕ_{ci} with a cumulative distribution function, Φ_{ci}, $\forall c = \{L, H\}$. The ξ_{ci} values uncorrelated and the expected values of ξ_{ci} vary depending upon the modelling situation. The majority of the assumptions stated for the pre-committed model are also applicable here.

In the literature, mostly two types of modelling approaches are used to analyse stochastic behaviour:

1 additive

2 multiplicative.

In the additive model, the stochastic demand is the sum of price-sensitive demand and a random factor. In the multiplicative model, the stochastic demand is calculated as a product of price-sensitive demand and its random factor. A more detailed overview of such modelling approaches with an application to Newsvendor pricing problem can be found in Petruzzi and Dada (1999).

4.1 Additive model

In the additive model, the random demand is the sum of price-sensitive demand and the random demand factor.

$$S_{ci} = D_{ci} + \xi_{ci}, \quad \forall c = \{L, H\}, i = \{1,2\}. \tag{17}$$

The random variable ξ_{ci} is drawn from two distinct distributions such that $E[\xi_{ci}] = 0$ and $\xi_{ci} \in [\underline{\xi}_{ci}, \overline{\xi}_{ci}]$. The payoff to airline i is $\Pi_i(B_i, \mathbf{P_L}, \mathbf{P_H})$, where $\mathbf{P_L} = (P_{Li}, P_{Lj})$ and $\mathbf{P_H} = (P_{Hi}, P_{Hj})$, $\forall\, i, j = \{1, 2\}$, $i \neq j$. For brevity, we write $\Pi_i = \Pi_i(B_i, \mathbf{P_L}, \mathbf{P_H})$.

Now, the total expected revenue generated by airline i offering only two fare classes can be written as shown in Equation (18). It is assumed that the demand arrival is sequential, i.e. low-fare class demand is observed before high-fare class demand, and thus a nested booking limit control is considered.

$$\Pi_i = P_{Li} \min\{S_{Li}, B_i\} + P_{Hi} \min\{S_{Hi}, C_i - \min\{S_{Li}, B_i\}\}. \tag{18}$$

The given revenue function, Equation (18), can further be partitioned for each fare class independently. Let Π_{Li} and Π_{Hi} be the revenues generated from low- and high-fare customers, respectively. Hence, the total expected revenue is:

$$\Pi_{Li} = E_{\xi_{Li}}[P_{Li} \min(S_{Li}, B_i)] = P_{Li}B_i - P_{Li}\int_{\underline{\xi}_{Li}}^{B_i - D_{Li}} \Phi_{Li}(\xi_{Li})d\xi_{Li} \tag{19}$$

assuming the condition, $\underline{\xi}_{Li} \le B_i - D_{Li} \le \overline{\xi}_{Li}$, is true.

Similarly, the expected revenue generated from a high-fare class, Π_{Hi} is

$$\begin{aligned}\Pi_{Hi} &= E_{\xi_{Hi}}E_{\xi_{Hi}}[P_{Hi}\min(S_{Hi}, C_i - \min(S_{Li}, B_i))] \\ &= P_{Hi}\left(C_i - B_i + \int_{\xi_{Hi}}^{Bi - D_{Li}} \Phi_{Li}(\xi_{Li})d\xi_{Li} - \int_{\xi_{Hi}}^{yi} \Phi_{Hi}(\xi_{Li})d\xi_{Hi}\right)\end{aligned} \tag{20}$$

where

$$y_i = C_i + \int_{\xi_{Hi}}^{Bi - D_{Li}} \Phi_{Li}(\xi_{Li})d\xi_{Li} - D_{Hi} - B_i$$

Assuming y_i is bounded as $\underline{\xi}_{Hi} \le y_i \le \overline{\xi}_{Hi}$. Now, the total expected revenue generated from both fare classes is:

$$\begin{aligned}\Pi_i = \Pi_{Li} + \Pi_{Hi} &= P_{Hi}C_i - (P_{Hi} - P_{Li})B_i + (P_{Hi} - P_{Li})\int_{\underline{\xi}_{Li}}^{B_i - D_{Li}} \Phi_{Li}(\xi_{Li})d\xi_{Li} \\ &- P_{Hi}\int_{\underline{\xi}_{Hi}}^{y_i} \Phi_{Hi}(\xi_{Hi})d\xi_{Hi}.\end{aligned} \tag{21}$$

In Equation (21), the first two terms are risk-free revenue gains that are observed when B_i seats are allocated for the low-fare class and the remaining seats are reserved for the high-fare class by airline i. The remaining two terms are the risk-involved revenue. As identified earlier in this model, we assumed that sequential demand arrival, which is the demand for low-fare class, is observed prior to the high-fare class. Therefore, the nested booking limit control is used in this model. The term, $(P_{Hi} - P_{Li})\int_{\underline{\xi}_{Li}}^{B_i - D_{Li}} \Phi_{Li}(\xi_{Li})d\xi_{Li}$, is the expected revenue gain when the demand for the low-fare class is less than the allocated capacity. It is assumed that the remaining capacity is used for the high-fare class. However in practice, it is unlikely to fill the extra capacity by the high-fare customers if the low-fare demand is lower than expected. Finally, the last term, $P_{Hi}\int_{\underline{\xi}_{Hi}}^{y_i} \Phi_{Hi}(\xi_{Hi})d\xi_{Hi}$, is the expected loss in revenue when airline i experiences a demand for its high-fare class which is less than the allocated capacity.

4.2 Multiplicative model

In the case of multiplicative model, demands in low and high-fare classes are modelled as follows:

$$S_{ci} = D_{ci}\xi_{ci}, \quad \forall c = \{L, H\}, i = \{1, 2\} \tag{22}$$

Similar to the additive model, the total expected revenue generated by airline i offering only two fare classes is again given by:

$$\begin{aligned}\Pi_i = \Pi_{Li} + \Pi_{Hi} = P_{Hi}C_i - (P_{Hi} - P_{Li})B_i + (P_{Hi} - P_{Li})D_{Li}\int_0^{B_i / D_{Li}} \Phi_{Li}(\xi_{Li})d\xi_{Li} \\ - P_{Hi}D_{Hi}\int_0^{k_i} \Phi_{Hi}(\xi_{Hi})d\xi_{Hi}\end{aligned} \tag{23}$$

where

$$k_i = C_i - B_i + D_{Li}\int_0^{B_i / D_{Li}} \Phi_{Li}(\xi_{Li})d\xi_{Li} / D_{Hi}$$

Assuming that $\underline{\xi}_{Hi} \le \frac{y_i}{D_{Hi}} \le \overline{\xi}_{Hi}$ holds, the Equation (23) has a similar structure as identified in aforementioned Equation (21).

5 Pricing and availability game: a numerical solution

Up to this point, we proposed two modelling approaches for the airline RM game. The first model assumes the space allocated for each fare class is known and firms compete for only through their pricing strategies. Pre-committed booking limits assumption enables us to analyse the model analytically which results unique pricing strategies at Nash equilibrium. The second approach uses stochastic demand, and the RM problem is modelled for both unknown pricing strategies and booking limits. Consequently, the problem becomes non-trivial to derive pricing and booking limit strategies analytically. Thus, a numerical analysis is suggested to study the fare-pricing competition along with a nested control on booking limits. The outcomes of the non-cooperative game studied numerically may result a unique Nash equilibrium for both pricing and availability, yet this is not proven in this paper.

The study focuses on the identification of pricing strategies and booking limits at Nash equilibrium. The developed computational model determines the optimal control parameters which includes low and high-fare prices, and low-fare booking limits for an airline. The (near) Nash equilibrium is searched numerically by using the following computational method for both additive and multiplicative models:

- *Step 1:* For each of the airlines the equilibrium is search, give $B_i \in [0, C_i]$, $P_{Li} \in [\underline{P}_{Li}, \overline{P}_{Li}]$, $P_{Hi} \in [\underline{P}_{Hi}, \overline{P}_{Hi}]$, $\forall i = \{1,2\}$.
- *Step 2:* Start at $B_2 = C_2/2$, $P_{L2} = (\underline{P}_{L2} + \overline{P}_{L2})/2$ and $P_{H2} = (\underline{P}_{H2}, \overline{P}_{H2})/2$.
- *Step 3:* Determine B_1, P_{L1}, and P_{H1} such that Π_1 (established in Equation (21) for additive model and Equation (23) for multiplicative) is maximised. A numerical procedure FMINCON in MATLAB is used to minimise $-\Pi_1$, and thus maximises Π_1.
- *Step 4:* Using the parameters determined in Step 3, maximise revenue to airline-2, Π_2, and determine B_2, P_{L2} and P_{H2}. Repeat Steps 3 and 4 sequentially to an extent when no airline is able to improve its payoff beyond an absolute value of 10^{-3}.

5.1 Symmetric market condition

The market demand randomness is assumed to be uniformly distributed. The price sensitive demand is modelled similar to the pre-committed booking limit case. In a standard symmetric game following, $C_i = 100$, $\underline{P}_{Li} = 0$, $\overline{P}_{Li} = 100$, $\underline{P}_{Hi} = 100$, $\overline{P}_{Hi} = 200$, $\alpha_{Li} = 60$, $\alpha_{Hi} = 40$, $\beta_{Li} = 0.25$, $\beta_{Hi} = 0.15$, $\forall i = \{1, 2\}$. Also $\theta_{Lij} = 0.15$, $\theta_{Hij} = 0.10$, $\forall i, j = \{1, 2\}, i \neq j$. For the additive model, the random demand factors, ξ_{Li} and ξ_{Hi} $\forall i = \{1, 2\}$ are bounded in $[\underline{\xi}_{Li}, \overline{\xi}_{Li}] = [-30, 30]$ and $[\underline{\xi}_{Hi}, \overline{\xi}_{Hi}] = [-30, 30], \forall i = \{1, 2\}$, respectively. For the multiplicative model, the random demand factors ξ_{Li} and ξ_{Hi}, $\forall i = \{1, 2\}$ bounded in $[\underline{\xi}_{Li}, \overline{\xi}_{Li}] = [0, 2]$ and $[\underline{\xi}_{Hi}, \overline{\xi}_{Hi}] = [0, 2]$, respectively.

The results of the numerical study are summarised in Table 3 for both additive and multiplicative models. Both airlines are facing a standard symmetric market conditions under the previously stated price and demand randomness. Because additive and multiplicative models are developed with distinct demand randomness criteria, they may result in closely related but different booking limits and fare pricing. We notice that the multiplicative model results in a higher payoff for both airlines along with higher parameters for controlling both the booking limits and the fare prices.

Table 3 A comparison of additive and multiplicative models

Model	*Airline*	*Booking limit*	*Low-fare price*	*High-fare price*	*Payoff*
Additive	Airline-1	72.35	176.53	205.18	13570.21
	Airline-2	72.35	176.53	205.18	13570.21
Multiplicative	Airline-1	84.90	175.50	208.32	13608.25
	Airline-2	84.90	175.50	208.32	13608.25

5.2 Impact study at asymmetric market condition

In this section, we extend the study in which the model related parameters for airline-2 are subject to variation and their impact is studied on payoff, booking limits and fare pricing of both competing airlines at Nash equilibrium. The suggested study uses Statistical Design of Experiment(s) (DOE) by considering a fractional factorial design with six factors each at two levels. The factors with their levels are identified in Table 4.

Table 4 Factors for two-level factorial design

Parameters	*Levels*
C_2	[80, 120]
β_{L2}	[0.2, 0.3]
β_{H2}	[0.1, 0.2]
θ_{L21}	[0.1, 0.2]
θ_{H21}	[0.05, 0.15]
Model type (I)	[Additive, multiplicative]

We used the DOE analysis to develop the first-order regression (Equation (24)). The equation is established by considering the factors that have a significant main and two-way interaction effect with a level of significance of 5% ($\alpha = 0.05$). When coded units are used in a regression equation then the factor at low level is replaced by –1 and similarly a factor at its high level is replaced by +1 (see Montgomery, 1991). The rest of the regression equations presented in this paper also use the same principle mentioned for development of Equation (24). From Equation (24), we conclude that an increase in the capacity of airline-2 (C_2) results a decrease in the payoff of airline-1. An increase of β_{L2} and β_{H2} for airline-2 also results a decrease in the payoff of airline-1 however, increasing θ_{H21} improves the payoff to airline-1. The interaction effect of β_{H2} and θ_{H21} is also significant. A simultaneous increase in β_{H2} and θ_{H21} result a decrease in the payoff of airline-1.

$$\hat{\Pi}_1 = 14861 - 598C_2 - 537\beta_{L2} - 1337\beta_{H2} + 867\theta_{H21} - 922\beta_{H2}\theta_{H21} \quad (24)$$

Similarly the DOE analysis is extended to study the payoff of competing airline-2. Later the booking limits and fare pricing are also studied for both airlines in a similar way. A DOE analysis with Π_2 as response has resulted in the first-order regression Equation (25). An increase in β_{L2} and β_{H2} decreases the payoff to airline-2 but an increase in θ_{L21} and θ_{H21} has opposite impact.

$$\hat{\Pi}_2 = 15234 - 1182\beta_{L2} - 3080\beta_{H2} + 1182\theta_{L21} + 2821\theta_{H21} - 1724\beta_{H2}\theta_{H21} \quad (25)$$

A regression analysis with booking limit B_1 as response is reported in Equation (26). An increase in β_{H2} and use of multiplicative model results in an increase in the booking limit to airline-1, B_1. A simultaneous increase or decrease in β_{H2} and θ_{H21} also increases the booking limit B_1.

$$\hat{B}_1 = 76.061 - 1.329\beta_{L2} + 1.019\beta_{H2} + 5.127I + 0.858\beta_{H2}\theta_{H21} \quad (26)$$

Like DOE analysis with B_1, a DOE analysis with B_2 as response has resulted in the regression reported in Equation (27). From Equation (27), we conclude that an increase in the capacity of airline-2 significantly increases its booking limit. Interaction effects $\beta_{H2}\theta_{H21}$ and $I\ \theta_{H21}$ also found to be significant. A simultaneous increase in factors β_{H2} and I results a reduction in B_2. The impact of simultaneous increase in the factors θ_{H21} and I causes an increase in B_2.

$$\hat{B}_2 = 69.998 + 10.033C_2 + 9.677\beta_{H2}\theta_{H21} - 7.946\beta_{H2}I + 7.605\theta_{H21}I \quad (27)$$

We also carried out DOE with the low-fare price of airline-1 and regression equation based on significant factors is shown in Equation (28). We can infer that an increase in capacity of airline-2, β_{L2} and β_{H2} significantly reduces the low-fare price P_{L1}. However, an increase in θ_{L21} and θ_{H21} results in an increase in P_{L1}.

$$\hat{P}_{L1} = 178.343 - 6.050C_2 - 7.156\beta_{L2} - 6.256\beta_{H2} + 4.449\theta_{L21} + 4.882\theta_{H21} \quad (28)$$

Equation (29) presents the first-order regression analysis from DOE in which the response in low-fare price offered by airline-2 is presented. An increase in C_2 and β_{L2} results in a significant decrease in P_{L2}. On the other hand, θ_{L21} is positively correlated to P_{L2}.

$$\hat{P}_{L2} = 169.00 - 13.63C_2 - 18.33\beta_{L2} + 10.83\theta_{L21} \quad (29)$$

Similarly, a DOE considering P_{H1} as response is presented using a regression in Equation (30). An increase in C_2 and β_{H2} results in a decrease in P_{H1} however, an increase in θ_{H21} and I results in a significant increase of P_{H1}. A simultaneous increase or decrease in β_{H2} and θ_{H21} results in a decrease in P_{H1}.

$$\hat{P}_{H1} = 235.63 - 6.83C_2 - 23.59\beta_{H2} + 13.78\theta_{H21} + 7.99I - 18.05\beta_{H2}\theta_{H21} \quad (30)$$

High-fare price P_{H2} decreases significantly with an increase in C_2 and β_{H2} as revealed in Equation (31). θ_{H21} and interaction of β_{H2} and θ_{H21} also have significant impact on P_{H2}.

$$\hat{P}_{H2} = 273.53 - 12.85C_2 - 54.36\beta_{H2} + 30.74\theta_{H21} - 40.48\beta_{H2}\theta_{H21} \quad (31)$$

5.3 Cross-effect analysis

In this section, a cross-effect analysis is presented. The analysis uses a standardised regression to study the effect of two control parameters practised by competing airlines on their revenue: fare pricing and booking limits.

In Table 5, a correlation matrix is presented by a factor analysis. The correlation coefficient explains the impact of one control parameter on the other parameter. For example, correlation between B_1 and B_2 is 0.55 which is also the standardised regression coefficient when only these two parameters are considered. Likewise, the correlation coefficient between P_{L1} and P_{L2} is 0.93, also the correlation coefficient between P_{H1} and P_{H2} is 0.988. Hence, we can conclude that a rise in booking limit and fare class prices also impacts the rise of these parameters for the competing airline.

Table 5 Correlation between parameters

Factors	B_1	P_{L1}	P_{H1}	B_2	P_{L2}	P_{H2}
B_1	1	–	–	–	–	–
P_{L1}	0.226	1	–	–	–	–
P_{H1}	0.046	0.719	1	–	–	–
B_2	0.550	–0.135	–0.277	1	–	–
P_{L2}	0.320	0.930	0.417	–0.002	1	–
P_{H2}	–0.059	0.672	0.988	–0.323	0.356	1

A factor analysis is also reported, and Table 6 presents the percentage of variation explained with a given number of factors. An analysis, using two factors only, is reported in Table 7. Using two factors approximately 81.1% of variation is explained. The two major factor loadings are on fare prices offered by an airline and its rival, as well as booking limits selection.

Table 6 Variation explained with an increase in the factors

Number of factors	*Initial eigen-values*			*Rotation sums of squared loadings*		
	Total	*Percentage of variance*	*Cumulative (%)*	*Total*	*Percentage of variance*	*Cumulative (%)*
1	3.138	52.305	52.305	3.135	52.243	52.243
2	1.728	28.797	81.102	1.732	28.860	81.102
3	0.755	12.591	93.693	–	–	–
4	0.372	6.203	99.897	–	–	–
5	0.006	0.097	99.993	–	–	–
6	0.000	0.007	100.000	–	–	–

Table 7 Two factors loading

	Factor 1	*Factor 2*
B_1	0.181	0.860
P_{L1}	0.950	0.164
P_{H1}	0.897	–0.228
B_2	–0.233	0.820
P_{L2}	0.773	0.373
P_{H2}	0.863	–0.321

6 Conclusions and future works

In this research work, we studied the airline RM game in a duopoly market to determine the competitive fare-pricing strategies and booking limits. Two scenarios are considered. In the first scenario, we showed the existence of a unique Nash equilibrium for pricing when the booking limits are pre-committed. In the second scenario, a joint determination of booking limits and fare pricing is addressed. Our main contributions in this research work are as follows:

1 Developed models that jointly determine nested booking limits and fare prices in a competitive duopoly market.

2 The fare-pricing game is analysed under various deterministic and stochastic demand conditions through sensitivity analysis. Results show that the developed mathematical models find unique booking limits and pricing strategies to competing airlines at the Nash equilibrium.

3 Parameters that have significant effects on the revenue are identified through a factorial design of experiment. Results also revealed unique regression equations for both pricing strategies and booking limits.

A direct extension to this work is to consider a cooperative game, i.e. the Nash Bargaining game under pre-committed booking limits and nesting strategies. This paper does not consider the dynamic version of the fare-price competition. The authors would like to advocate the potential benefits of dynamically updating fare prices in today's competitive environment. Neuro-dynamic programming is a good tool to study the dynamic version of fare-pricing competition in a single flight leg and also in flight network settings.

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Appendix

Proof of Proposition 1: We prove this argument in duopoly competition. Let us consider the revenue of airline-1 as:

$$\Pi_1 = P_{L1}B_1 - P_{L1}[B_1 - D_{L1}]^+ + P_{H1}(C_1 - B_1) - P_{H1}[C_1 - B_1 - D_{H1}]^+ \quad (32)$$

Once the booking limit is known, the total revenue function is decomposed to revenue generated from low-fare class (Π_{L1}), and high-fare class (Π_{H1}), such that

$$\Pi_{L1} = P_{L1}B_1 - P_{L1}[B_1 - D_{L1}]^+ \quad (33)$$

$$\Pi_{H1} = P_{H1}(C_1 - B_1) - P_{H1}[C_1 - B_1 - D_{H1}]^+ \quad (34)$$

Hence, Π_{L1} is supermodular function in (P_{L1}, P_{L2}). The similar analogy is used to draw the same conclusion for Π_{H1}.

$$\frac{\partial \Pi_{L1}}{\partial P_{L1}} = B_1 - [B_1 - D_{L1}]^+ - P_{L1}\frac{\partial [B_1 - D_{L1}]^+}{\partial P_{L1}} \quad (35)$$

$$\frac{\partial^2 \Pi_{L1}}{\partial P_{L1}\partial Z_{L2}} = -\frac{\partial [B_1 - D_{L1}]^+}{\partial Z_{L2}} - P_{L1}\frac{\partial^2 [B_1 - D_{L1}]^+}{\partial P_{L1}\partial Z_{L2}} \quad (36)$$

Form assumption 2′, $\frac{\partial D_{L1}}{\partial Z_{L2}} < 0$, thus $\frac{\partial [B_1 - D_{L1}]^+}{\partial Z_{L2}} \geq 0$. Also from assumptions 2′ and 3′, we know that $B_1 - D_{L1}$ is increasing in both P_{L1} and Z_{L2}, hence it is a submodular function. Moreover, as $[x]^+ = \max\{x, 0\}$ is a convex increasing function. Therefore, by using Lemma 3, we can conclude that $\frac{\partial^2 [B_1 - D_{L1}]^+}{\partial P_{L1}\partial Z_{L2}} \geq 0$, and we obtain $\frac{\partial^2 \Pi_{L1}}{\partial P_{L1}\partial Z_{L2}} \leq 0$ which proofs that it is a supermodular function in (P_{L1}, P_{L2}).

Since the revenue function of each airline is supermodular, a unique Nash equilibrium exists if $\left|\frac{\partial^2 \Pi_{L1}}{\partial P_{L1}^2}\right| \geq \left|\frac{\partial^2 \Pi_{L1}}{\partial P_{L1}\partial Z_{L2}}\right|$

(Topkis, 1979) where:

$$\left|\frac{\partial^2 \Pi_{L1}}{\partial P_{L1}^2}\right| = \left|2\frac{\partial [B_1 - D_{L1}]^+}{\partial P_{L1}} + P_{L1}\frac{\partial^2 [B_1 - D_{L1}]^+}{\partial P_{L1}^2}\right| \quad (37)$$