[3] R. Kelly, "Comments on adaptive PD controller for robot manipulators," *IEEE Trans. Robot. Automat.*, vol. 9, pp. 117–119, Feb. 1993.
[4] S. Arimoto, *Control Theory of Nonlinear Mechanical Systems—A Passivity-Based and Circuit-Theoretic Approach*. Oxford, U.K.: Clarendon, 1996.
[5] J. J. E. Slotine and W. Li, "Adaptive manipulator control: A case study," *IEEE Trans. Automat. Contr.*, vol. 33, pp. 995–1003, Nov. 1988.
[6] C. C. Cheah and D. Wang, "Learning impedance control for robotic manipulator," *IEEE Trans. Robot. Automat.*, vol. 14, pp. 452–465, June 1998.
[7] F. Miyazaki and Y. Masutani, "Robustness of sensory feedback control based on imperfect Jacobian," in *Proc. Robotic Research: 5th Int. Symp.*, 1990, pp. 201–208.
[8] R. Kelly, "Global positioning of robot manipulator via pd control plus a class of nonlinear integral actions," *IEEE Trans. Automat. Contr.*, vol. 43, pp. 934–938, July 1998.
[9] R. Kelly, "Regulation of manipulators in generic task space: An energy shaping plus damping injection approach," *IEEE Trans. Robot. Automat.*, vol. 15, pp. 381–386, Apr. 1999.
[10] R. Kelly, R. Carelli, O. Nasisi, B. Kuchen, and F. Reyes, "Stable visual servoing of camera-in-hand robotic systems," *IEEE/ASME Trans. Mechatron.*, vol. 5, pp. 39–48, Jan. 2000.
[11] C. C. Cheah, S. Kawamura, and S. Arimoto, "Feedback control for robotic manipulators with an uncertain jacobian matrix," *Int. J. Robot. Syst.*, vol. 12, no. 2, pp. 119–134, 1999.
[12] C. C. Cheah, M. Hirano, S. Kawamura, and S. Arimoto, "Approximate jacobian control for robots with kinematics and dynamics," *IEEE Trans. Robot. Automat.*, vol. 19, pp. 692–702, Aug. 2003.
[13] H. Yazarel and C. C. Cheah, "Task-space adaptive control of robotic manipulators with uncertainties in gravity regressor matrix and kinematics," *IEEE Trans. Automat. Contr.*, vol. 47, pp. 1580–1585, Sept. 2002.
[14] S. Arimoto, "Robotics research toward explication of everyday physics," *Int. J. Robot. Res.*, vol. 18, no. 11, pp. 1056–1063, 1999.
[15] C. C. Cheah, K. Li, S. Kawamura, and S. Arimoto, "Approximate jacobian control of robot and its passability at singular points," in *Proc. Int. Conf. Control, Automation, Robotics, Vision*, Singapore, 2000.
[16] Hamamatsu Photonics K.K. [Online]. Available: http://www.hamamatsu.com/

Stabilization of Uncertain Nonholonomic Systems via Time-Varying Sliding Mode Control

Yueming Hu, Shuzhi Sam Ge, and Chun-Yi Su

***Abstract*—This note addresses the robust stabilization problem for a general class of nonholonomic systems in the presence of drift uncertainties. The control approach developed is based on the combined applications of the sliding mode control technique and nonlinear time-varying systems theory. First, some properties of nonlinear time-varying systems are introduced for the purpose of designing sliding mode controller. An explicit time-varying feedback form is provided to guarantee the existence and uniqueness of periodic time-varying solution for the corresponding linear periodic partial differential equation. Second, an explicit discontinuous feedback control law is presented to guarantee the existence of sliding mode. The first integrals obtained by the previous periodic partial differential equation are then directly used to determine the switching function. The uniform asymptotic stability of the closed loop system is proved via the invariance principle of nonlinear time-varying systems. Finally, an example is given to illustrate the proposed approach.**

***Index Terms*—Nonholonomic systems, robustness, stabilization, time-varying state feedback.**

I. Introduction

Control of nonholonomic systems has been an active field of research for a decade. Such systems can be found frequently in mechanical systems such as wheeled mobile robots, car-like vehicle, knife-edge, and so on (see, for example, [4], [11], and [12]). It is well known that nonholonomic systems, although controllable, cannot be stabilized by any time-invariant continuous state feedback control law (see [11], and the references therein). This fact makes the control of general nonholonomic systems extremely challenging, and stimulates researchers to construct time-varying or discontinuous feedback controllers for the control of nonholonomic systems. Many elegant control strategies, such as smooth time-varying strategies in [10], [12], [14], [16], [18], and [19], discontinuous control laws in [1]–[3], and [10], and the combined strategies of the two [20], time optimal control law in [17], iterative learning control in [15], and feedback linearization [6] have been proposed for various nonholonomic systems. However, most of current control approaches are developed for driftless nonholonomic systems, especially for a special class of systems in the so-called chained form, which was brought to the literature in [14]. Furthermore, they need complete knowledge of the systems.

An important issue for a practical system design is the robustness consideration against possible modeling errors and external disturbances, i.e., the systems with drift uncertainties. However, the latter case has received relatively less attention. The design of robust controllers for general nonholonomic systems with drift uncertainties is thus emphasized in the recent survey [11] as well as in the paper [9]. It is therefore our belief that it is timely to address the control issue for the systems with drift uncertainties by currently available controller design techniques.

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Sliding mode control (SMC) is a special discontinuous control technique applicable to a broad variety of practical systems (see, for instance, [22]). This approach is mainly based on the design of switching functions of state (or output) variables, which are used to create a sliding manifold. When this manifold is attained, the switching functions keep the trajectory on the manifold, thus yielding desired system dynamics. Due to its easy implementation and good robustness, SMC is considerably attractive for many highly nonlinear uncertain systems. Generally speaking, if the classical Pfaffian constraints generated by the input matrix are completely integrable, then the sliding mode subsystem of a smooth nonlinear affine control system can be easily reduced to a lower order system [22]. Thus, the customary design methods can be used to determine the sliding manifold. However, such design technique is no longer available for nonholonomic control systems since the related Pfaffian equations in this situation do not offer any nontrivial solutions, which have to be used in the construction of the necessary state transformation.

Since any time-invariant continuous state feedback control law cannot stabilize nonholonomic systems, the SMC technique becomes appealing in the control of nonholonomic systems. So far, only a few researchers have investigated the design of sliding mode controllers for a class of nonholonomic systems, which heavily depend on the existence of suitable Lyapunov functions and special structures of the systems (see, for instance, [3], [7], [16], and [21]). Even if the necessary Lyapunov function is constructed, the resulted sliding mode motion is only locally ensured. Thus the problem of designing sliding mode controllers for a general class of nonholonomic systems with drift uncertainties is still a challenging task and remains a subject of future research.

This note addresses the robust control problem for general nonholonomic systems with drift uncertainties whioch may not be limited to the so-called chained form. Motivated by the fact that nonholonomic systems can be asymptotically stabilized by time-varying feedback laws or discontinuous control laws, we develop a robust control approach based on both SMC theory [22] and nonlinear time-varying systems theory ([5] and [8]). Apart from the robustness against uncertainties, one important feature of our approach is that only the solutions of prescribed solvable initial value problem (IVP), which provides an explicit time-varying feedback law, are required in the design of the sliding mode controller.

The rest of the note is organized as follows. Section II presents the preliminary concepts and some properties of nonlinear time-varying systems involved in the design of sliding mode controllers. An explicit time-varying feedback form is presented to guarantee the existence and uniqueness of periodic time-varying solution for the IVP, or equivalently, nth independent first integrals for the corresponding linear periodic partial differential equation (PDE). Section III focuses on the design of time-varying sliding mode controllers. In the presence of uncertainties, a time-varying SMC law is proposed to force the trajectories of a system attaining to a time-varying sliding manifold. It is shown that, under the matching condition, the sliding motion is invariant with respect to uncertainties. The uniform asymptotic stability of the sliding motion is proved by using the invariance principle [5]. Finally, the proposed approach is applied to a hopping robot in flight phase to illustrate the proposed method.

II. Technical Preliminaries

Consider the following nonlinear control system:

$$\dot{x} = B(x)u + D(t,x) \tag{1}$$

where $x \in R^n$ and $u \in R^m$ are the state and input vectors respectively; $B = (b_1, \ldots, b_m) \in R^{n\times m}$ with $\{b_i\}$ being sufficiently smooth and linearly independent vector fields defined on an open subset $\Omega \subset R^n$ (the arguments of functions or distributions will be often suppressed for simplicity in the following if no confusion is caused); $D(t,x)$ represents drift uncertainties, which may include unmodeled dynamics and parameter variations. Without $D(t,x)$, the corresponding system becomes the so-called drift free system. System (1) can also be treated as nonholonomic systems with drift uncertainties, which appears in many mechanical plants such as mobile robots [12] and [15].

The aim of this note is to stabilize the system (1) at a desired point $x_d \in \Omega$ via time-varying SMC of the form

$$u = \alpha(t,x) + \beta(t,x)\mathrm{sign}\,(S(t,x)) \tag{2}$$

where $\alpha(t,x) \in R^m$, $\beta(t,x) \in R^{m\times m}$, and sliding manifold $S(t,x) \in R^m$ are proper time-varying smooth functions to be determined in the subsequent sections and $\mathrm{sign}(.)$ denotes the signum function vector. In the following development, we only consider the case that $x_d = 0 \in \Omega$. If it is not the case, a transformation $\zeta = x - x_d$ is necessary to apply the developed method.

Let $\Delta = span\{b_1, \ldots, b_m\}$ be the distribution associated with the system (1), and define the iterative distributions Δ_i as follows:

$$\Delta_1 = \Delta \quad \Delta_i = \Delta_{i-1} + [\Delta_1, \Delta_{i-1}], \qquad i = 2, 3, \ldots$$

where $[\Delta_1, \Delta_{i-1}] = span\{[f,g] : f \in \Delta_1, g \in \Delta_{i-1}\}$, and [.,.] is the Lie bracket operator. We will assume the following.

A1) $\Delta_i(x) (i = 2, 3, \ldots)$ are regular for all $x \in \Omega$.

A2) $\mathrm{rank}\ \Delta_{r+1} = n$ for all $x \in \Omega$, where r is the degree of nonholonomy of the distribution Δ.

A3) There is a sufficiently smooth ω-periodic function $p(t,x)$: $R \times \Omega \to R^m$ with $p(t,0) = 0$ such that the IVP

$$\begin{cases} \dot{x} = B(x)p(t,x) \\ x(t_0) = x_0 \end{cases} \tag{3}$$

has a unique ω-periodic solution $x = x(t, t_0, x_0)$, which is continuously differentiable with respect to all of its variables.

Assumptions A1) and A2) are essential to guarantee the controllability of (1). While Assumption A3) implies that (3) is completely integrable and its flow is periodic (see, [5] and [8]). Under certain conditions an explicit form of $p(t,x)$ was presented in [16]. Proposition 1 gives more general explicit conditions to satisfy Assumption A3).

Proposition 1: Let $p(t,x) = \varepsilon\chi(t,x), \chi(t,x)$: $R \times \Omega \to R^m$ be any smooth function such that $\chi(t+\omega, x) = \chi(t,x)$ and $\chi(-t,x) = -\chi(t,x)$, for all $(t,x) \in R \times \Omega$.

Then, there exists $\varepsilon_0 > 0$ such that Assumption A3) is satisfied and $x(t_0 - t, t_0, x_0) = x(t - t_0, t_0, x_0)$ for all $\varepsilon \in (0, \varepsilon_0]$ and for all $t \in R$.

Proof: See the Appendix.

The above proposition delineates the explicit time-varying feedback form guaranteeing the existence of periodic solution with respect to the IVP (3). It evidently generalizes the previous results in [16] since any sufficiently smooth periodic odd function can generate the periodic Lyapunov function needed in the design of feedback controller [16].

With this in mind, denote a gradient function for any smooth multivariable function $\xi(\tau)$ as $\nabla_\tau \xi = (\partial\xi/\partial\tau)$, the following lemma can be established.

Lemma 1: Under Assumption A3), the linear PDE

$$\nabla_t z + (\nabla_x z)B(x)p(t,x) = 0 \tag{4}$$

has n sufficiently smooth solutions $z = z(t,x) = (z_1(t,x), \ldots, z_n(t,x))^T = x(t_0, t, x_0)$ such that $\nabla_x z$ is nonsingular for all $(t,x) \in [0,\infty) \times \Omega$. Conversely, if the linear PDE (4) has n sufficiently smooth solutions $z = z(t,x)$ such that $\nabla_x z$ is nonsingular for all $(t,x) \in [0,\infty) \times \Omega$ and let $x = x(t,z)$ be the inverse function of $z = z(t,x)$, then $x = x(t, z(t_0, x_0))$, which is differentiable with respect to all of its variables, is the unique solution of (3). Furthermore, let $\eta(t,x)$ be a differentiable function, then $\eta(t,x)$ is a first integral of (3) if and only if there is a differentiable function $K = K(z)$ such that $\eta(t,x) = K(z(t,x))$. Besides, for all $(t,x) \in [0,\infty) \times \Omega$, the above first integrals $z(t,x)$ of (3) satisfy

$$z(t+\omega, x) = z(t,x) \quad z(t_0, x) = x \tag{5}$$

$$V = \frac{1}{2} z^T(t,x) z(t,x) = 0, \qquad \text{if and only if } x = 0. \tag{6}$$

Proof: The conclusion can be directly obtained by [8, Th. 12] and is similar to the proof in [16].

Thus, the existence and uniqueness of sufficiently smooth periodic solution of the IVP (3) are equivalent to that of n independent periodic solutions of linear PDE (4).

Remark: Particularly, let $\lambda(t,x)$ be any sufficiently smooth ω-periodic even function with respect to t such that $(\nabla_x \lambda B)^{-1} \nabla_t \lambda$ is smooth enough and $\mathrm{rank}\,(\nabla_x \lambda B) = m$, then

$$\chi(t,x) = -\varepsilon^{-1} (\nabla_x \lambda B)^{-1} \nabla_t \lambda, \qquad \varepsilon > 0 \tag{7}$$

is a ω-periodic odd function satisfying the conditions in Proposition 1. Hence, with the selection (7), the assigned $\lambda(t,x)$ should satisfy the PDE (4), that is, $\lambda(t,x)$ is an independent m-dimensional solution vector of (4). This fact shows that the linear PDE can easily be reduced to find only $n-m$ independent solutions of (4) since $\lambda(t,x)$ can be properly assigned.

Since $\mathrm{rank}\, B = m$, without loss of generality we assume that x, B and S are represented as

$$B(x) = \begin{pmatrix} B_1(x_1, x_2) \\ B_2(x_1, x_2) \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad S = S(t, x_1, x_2) \tag{8}$$

where $x_1 \in R^m$, $x_2 \in R^{n-m}$; $B_1 \in R^{m\times m}$ with $\mathrm{rank}\, B_1 = m$; and $B2 \in R^{(n-m)\times m}$. Let $B_0 = B_2 B_1^{-1} \in R^{(n-m)\times m}$ be smooth enough, then we have the following proposition.

Proposition 2: Let $\phi(t) : R \to R^{m\times m}$ be a smooth ω-periodic matrix function such that

$$\phi(t+\omega) = \phi(t) \; and \; \phi(-t) = -\phi(t) \qquad \forall t \in R. \tag{9}$$

Then, the IVP (3) can be reduced to the following $(n-m)$-dimensional IVP

$$\begin{cases} \dot{x}_2 = \varepsilon B_0 \left(Q(t,t_0)x_{10}, x_2\right) \phi(t) Q(t,t_0) x_{10} \\ x_2(t_0) = x_{20} \end{cases} \tag{10}$$

and Assumption A3) is satisfied for $\varepsilon \in (0, \varepsilon_0]$, where $Q(t,t_0) = \exp\{\varepsilon \int_0^t \phi(\tau) d\tau\}$ is an even ω-periodic matrix function, and $\varepsilon_0 > 0$ is sufficiently small.

Proof: See the Appendix.

The aforementioned propositions show that the first integrals used in the design of switching function can be explicitly obtained for a wide class of periodic feedback odd function $p(t,x)$. Particularly, for the decomposition (8) and $p(t,x) = \varepsilon \phi(t) x_1$, we only need to solve a reduced order (actually $(n-m)$-dimensional) IVP.

III. Controller Design via Time-Varying Sliding Mode Technique

As indicated in [22], the design of controller via sliding mode technique consists of two steps. First, a switching manifold is determined to ensure that the sliding motion does have a good dynamic response. Second, a proper discontinuous feedback control law is designed to achieve the desired sliding mode.

A. Switching Function Design Based on the First Integrals

We first design a proper switching function $S(t,x)$ so that the resulted sliding motion is asymptotically stable. Based on the results developed in [22], the solution can be obtained as follows.

Proposition 3: Under Assumptions A1)–A3), the sliding mode system is locally uniformly asymptotically stable if $S = S(t,x)$ is chosen to be the form

$$S(t,x) = B^T (\nabla_x z)^T z \tag{11}$$

where $z = z(t,x)$ are n independent first integrals of (4). The auxiliary smooth function $p(t,x)$ vanishes only for $x = 0$ and satisfies Assumption A3).

Proof: The proof of the proposition is given in the Appendix.

This method of designing a sliding mode controller only involves n independent solutions of a prescribed first-order linear periodic PDE, which is always solvable for a wide class of $p(t,x)$ as indicated by the Propositions 1 and 2.

B. Discontinuous Control Law

The next step is to choose $\alpha(t,x) \in R^m$ and $\beta(t,x) \in R^{m\times m}$ for the control law (2) so that the trajectories of the closed-loop system (1) with control (2) realize the sliding motion in finite time.

Proposition 4: If $C \overset{\text{def}}{=} (\nabla_x S)B$ is designed to be nonsingular, and the uncertainty $D(t,x)$ in (1) is bounded by a known function $d(t,x)$ such that $|D(t,x)| \le d(t,x)$, then using the following discontinuous control law:

$$u = -C^{-1} \left[\nabla_t S + K_1 S + K_2 \mathrm{sign}(S)\right] \tag{12}$$

the trajectories of the closed-loop system (1) will reach the sliding manifold $S(t,x) = 0$ defined in (11) in finite time, where

$$\begin{cases} K_i = \mathrm{diag}(k_{ij})_{m\times m}, & i = 1,2;\; j = 1,\ldots,m \\ k_{1j} \ge 0;\; \min_{1\le j\le m}\{k_{2j}\} & \\ \quad \ge |\nabla_x S| d(t,x) + k_{20}, & k_{20} = \mathrm{const} > 0 \end{cases} \tag{13}$$

with the norm $|.|$ of vector or matrix defined by $|(a_{ij})_{p\times q}| = \sum_{i=1}^p \sum_{j=1}^q |a_{ij}|$.

Proof: From (1), (12), and (13), one can obtain

$$\dot{S} = \nabla_t S + \nabla_x S(Bu + D) = \nabla_x S D - K_1 S - K_2 \mathrm{sign}(S) \tag{14}$$

which by (14), implies that

$$\begin{aligned} S^T \dot{S} &\le -S^T K_1 S - S^T K_2 \mathrm{sign}(S) + |S| |\nabla_x S| |D(t,x)| \\ &\le -S^T K_1 S - k_{20}|S| < 0. \end{aligned} \tag{15}$$

That is, the reaching condition $S^T \dot{S} < 0$ of the sliding mode is satisfied [22].

Therefore, the discontinuous control law (12) can easily be determined once the bounds of system uncertainties or parameter disturbances are known. The condition that C is nonsingular ensures the system has the regular sliding mode, thus, the dynamic equation of sliding motion can be directly derived by the equivalent control principle [22]. While the purpose of the term $-C^{-1} K_1 S$ in the control law (12) is to lessen the chattering phenomenon exhibited by high frequency vibration of the controlled plant.

It is important to note that the discontinuous control law (12) is used to force any trajectory of (1) to reach the sliding manifold, which is just the invariant manifold discussed in [16], and thus guarantees uniform asymptotic stability of the closed-loop system. The global asymptotic stability can even be guaranteed if $p(t,x)$ is chosen such that the resulted matrix C is nonsingular globally, which will be illustrated through an example in Section IV.

C. Sliding Mode Equations

Once the sliding motion is attained, from (14) we can obtain the equivalent control u_{eq} as

$$u_{eq} = -C^{-1}\left[\nabla_t S + (\nabla_x S)D\right]. \quad (16)$$

Substituting (16) into (1) yields the following sliding mode equation:

$$\begin{cases} S(t,x) = 0 \\ \dot{x} = -BC^{-1}\nabla_t S + D - BC^{-1}(\nabla_x S)D \end{cases}. \quad (17)$$

From (17), it is obvious that the switching functions $S(t,x)$ must be time-varying; otherwise the stabilization cannot be achieved. Particularly, if the uncertain part $D(t,x)$ in (1) satisfies the matching condition $\mathrm{rank}(B,D) = \mathrm{rank}B$, then the sliding mode equation (17) is reduced to the form

$$\begin{cases} S(t,x) = 0 \\ \dot{x} = -BC^{-1}\nabla_t S \end{cases} \quad (18)$$

that is, the sliding motion is invariant with respect to $D(t,x)$. (18) shows that only time-varying switching function can produce asymptotically stable sliding motion for (1).

IV. Example

In this section, we will concentrate on the development of a sliding mode controller for the hopping robot in the flight phase in [12]. The configuration $q = (\psi, l, \theta)$ consists of the leg angle, the leg extension, and the body angle of the robot. Let I and m represent, respectively, the moment of inertia of the body and the mass of the leg concentrated at the foot, let the upper leg length be taken as d, with l representing the extension of the leg past this point.

Considering the conservative constraint of the angular momentum, one can obtain the control equations as follows [12]:

$$\dot{\psi} = u_1 \quad \dot{l} = u_2 \quad \dot{\theta} = -\left\{\frac{m(l+d)^2}{[I+m(l+d)^2]}\right\}u_1. \quad (19)$$

Different approaches have been proposed by Taylor series expanding about $l = 0$ and nonlinear state transformation [12]. However, due to the uncertainties of the parameters I and m, one needs to consider the robustness of the control against parameter uncertainties. Let M_0 and ΔM be the certain and uncertain parts of $\{m(l+d)^2/[I+m(l+d)^2]\}$, then (19) can be rewritten as follows:

$$\dot{\psi} = u_1 \quad \dot{l} = u_2 \quad \dot{\theta} = -(M_0 + \Delta M)u_1 \quad (20)$$

which is in the form of (1), and the uncertain part does not satisfy the invariance condition of sliding mode [22]. Let $p(t,x) = \varepsilon sint(\theta, 0)^T (\varepsilon > 0)$ in A3), by technical calculations, we obtain the periodic solution of the corresponding IVP (3) for the nominal part of (20) as

$$\begin{aligned} \psi(t) &= \psi(t_0) + \theta(t_0)\left[1 - \exp\{\varepsilon(\mathrm{cos}t - \mathrm{cos}t_0)\}\right] \\ l(t) &= l(t_0), \qquad \theta(t) = \theta(t_0)\exp\{\varepsilon(\mathrm{cos}t - \mathrm{cos}t_0)\} \end{aligned} \quad (21)$$

which is 2π-periodic for any constant $\varepsilon > 0$. The first integrals are thus, by Lemma 1, given as

$$z_1(t) = \psi + \theta(1-Y) \quad z_2(t) = l \quad z_3(t) = Y\theta \quad (22)$$

where $Y = \exp\{\varepsilon\alpha\}$, $\alpha = cost_0 - cost$. Thus, by (11), one obtain the corresponding time-varying switching function

$$S = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} [\psi + (1-2Y)\theta]\,Y \\ l \end{pmatrix} \quad (23)$$

and control law

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = -\begin{pmatrix} \frac{1}{2}Y^{-1}\varepsilon\mathrm{sin}t(\theta+\psi) - 2\varepsilon\mathrm{sin}t\theta \\ +\frac{1}{2}Y^{-2}\left[k_{11}s_1 + k21\mathrm{sign}(s_1)\right] \\ k_{12}s_2 + k_{22}\mathrm{sign}(s_2) \end{pmatrix} \quad (24)$$

with

$$\begin{aligned} C &= (\nabla_x S)B = \begin{pmatrix} 2Y^2 & 0 \\ 0 & 1 \end{pmatrix} \\ \frac{\partial S}{\partial t} &= \begin{pmatrix} \varepsilon\mathrm{sin}tY(\theta+\psi) - 4\varepsilon\mathrm{sin}tY^2\theta \\ 0 \end{pmatrix} \end{aligned} \quad (25)$$

where C is globally invertible. Since

$$\dot{S} = \begin{pmatrix} \dot{s}_1 \\ \dot{s}_2 \end{pmatrix} = \begin{pmatrix} \varepsilon\mathrm{sin}tY(\theta+\psi) - 4\varepsilon\mathrm{sin}tY^2\theta \\ +\left[2Y^2 + (2Y-1)Y\Delta M\right]u_1 \\ u_2 \end{pmatrix}. \quad (26)$$

For any $k_{i2} > 0$ $(i = 1,2)$, with the controller (24), one has $\dot{s}_2 = -k_{12}s_2 - k_{22}\mathrm{sign}(s_2)$, which implies that the sliding motion $s_2 = 0$ must occur in finite time [22].

Based on (15), k_{i1} $(i = 1,2)$ can be selected as

$$\begin{aligned} &k_{11} > 0 \\ &k_{21} > \frac{1}{2(1-\hat{M})}\left|(2Y-1)[\varepsilon sint(\theta+\psi) - 2\varepsilon sint\theta]\right|\hat{M} \end{aligned} \quad (27)$$

it can be confirmed that the sliding motion $s_1 = 0$ occurs in finite time [22].

Finally, let us examine the robustness once the sliding mode is achieved. In the sliding mode, the general solutions can be obtained as follows:

$$\psi(t) = \psi(t_r) - \frac{2+\Delta M}{(1+\Delta M)\left[2Y + (2Y-1)\Delta M\right]}\theta(t_r) \quad (28)$$

$$\theta(t) = \frac{(2+\Delta M)}{\left[2Y + (2Y-1)\Delta M\right]}\theta(t_r) \quad (29)$$

where t_r is the reaching time of the sliding mode, which implies that $\theta(t_r) = 0$ and $\psi(t_r) = 0$. Therefore, once the sliding mode is realized, the motion is independent of uncertainty ΔM. This example shows the robustness of the proposed approach against mismatched uncertainty.

V. Conclusion

In this note, a sliding mode controller design technique has been developed for a general class of nonholonomic systems with drift uncertainties. Explicit approaches have been developed for designing the switching functions from the solutions of prescribed linear PDE. The proposed control law can be determined from the possible bounds of the uncertainties.

Appendix
Proof of Proposition 1

First, we consider the following periodic system:

$$\dot{x} = g(t) \quad (30)$$

where $g(t): R \to \Omega \subset R^n$ is a continuous and ω-periodic function. Define the subspace

$$P_\omega = \{g \in C(R,\Omega) | g(t+\omega) = g(t)\}, \qquad \|g\| = sup_{t\in[0,\omega]}|g(t)| \quad (31)$$

and the mapping $P: P_\omega \to P_\omega$ as follows:

$$(Pg)(t) = \frac{1}{\omega}\int_0^\omega g(\tau)d\tau, \qquad g \in P_\omega. \quad (32)$$

Then

$$\|Pg\| \leq \|g\|, \text{ i.e., } \quad \|P\| \leq 1. \tag{33}$$

Hence, there exists a unique ω-periodic solution for (30) if and only if $(Pg)(t) = 0$. Once $(Pg)(t) = 0$, then the solution $x(t)$ of (30) must satisfy $(Px)(t) = 0$. Denote the unique ω-periodic solution as $x(t) = (Kg)(t)$, then

$$(Kg)(t) = (I - P) \int_0^t g(\tau) d\tau$$
$$(I \text{ is the identity maping}), \qquad \|Kg\| \leq 2\omega\|g\|. \tag{34}$$

Besides, if $g(-t) = -g(t)$, then $(Kg)(-t) = (Kg)(t)$.

Let $f(t, x) = B(x)\chi(t, x)$, then the IVP (3) in this case is equivalent to

$$\begin{cases} \dot{x} = \varepsilon f(t, x) \\ x(t_0) = x_0 \end{cases} \tag{35}$$

where $f(t + \omega, x) = f(t, x)$ and $f(-t, x) = -f(t, x)$, $\forall (t, x) \in R \times \Omega$ since $\chi(t, x)$ is an odd periodic function by the assumptions. Only $t_0 = 0$ will be considered in the following due to the periodicity of (35).

Consider $g(t) = f(t, x(t))$ in (30), it is not difficult to prove that (35) exists periodic solution $x(t)$ if and only if

$$\begin{cases} x(t) = x_0 + \varepsilon K(I - P) f(t, x(t)) \\ Pf(t, x(t)) = 0 \end{cases}. \tag{36}$$

To prove the solution of (35) is ω-periodic, we will use the fixed point theorem of contraction mapping. For any $\beta > 0$, define the following set

$$B(\beta) = \{y \in P_\omega | y(-t) = y(t), \|y\| \leq \beta, (Py) = 0\}. \tag{37}$$

Then, $B(\beta)$ is a closed convex subset of P_ω.

Consider the mapping $Q : B(\beta) \to P_\omega$ defined as follows:

$$(Qy)(t) = \varepsilon K(I - P) f(t, y(t) + x_0)$$
$$y \in B(\beta), \qquad x_0 \in B_\alpha = \{x \in \Omega \| x| \leq \alpha\}. \tag{38}$$

Let

$$F_1 = sup_{t \in [0,\omega], |x| \leq \alpha + \beta} |f(t, x)|$$
$$F_2 = sup_{t \in [0,\omega], |x| \leq \alpha + \beta} \left| \frac{\partial f(t, x)}{\partial x} \right|$$

and select $\varepsilon_0 > 0$ such that

$$4\varepsilon_0 \omega F_1 < \beta, \qquad 4\varepsilon_0 \omega F_2 < \frac{1}{2} \tag{39}$$

i.e., $0 < \varepsilon_0 < min\{\beta/4\omega F_1, 1/8\omega F_2\}$. Then, we have

$$(Qy)(-t) = (Qy)(t) \quad \|Qy\| \leq \beta \quad P(Qy)(t) = 0$$
$$\forall \varepsilon \in (0, \varepsilon_0], \qquad x_0 \in B_\alpha, \qquad y \in B(\beta). \tag{40}$$

Since $f(t, x)$ is an odd function with respect to t, (39) implies that $Qy \in B(\beta)$.

Furthermore, from (38) it follows that

$$|(Qy_1)(t) - (Qy_2)(t)| = |\varepsilon K(I - P)[f(t, y_1(t) + x_0)$$
$$- f(t, y_2(t) + x_0)|, \qquad y_1, y_2 \in B(\beta). \tag{41}$$

Based on the norm bounds (33) and (34) for the mapping K and P, (41) implies that

$$\|Qy_1 - Qy_2\| \leq 4\varepsilon\omega F_2 \|y_1 - y_2\| \leq \frac{1}{2} \|y_1 - y_2\|. \tag{42}$$

That is, Q is a contraction mapping from $B(\beta)$ to P_ω. By the well-know fixed point theorem [6], [10], there exists unique fixed point $y^* = y^*(x_0, \varepsilon) \in B(\beta)$ for $x_0 \in B_\alpha$ and $\varepsilon \in (0, \varepsilon_0]$.

Finally, let $x^* = x^*(x_0, \varepsilon) = y^* + x_0$, then x^* is the solution of (35), satisfying $x^*(-t) = x^*(t)$ due to $y^* \in B(\beta)$. To prove the periodicity of the solution, we only need to prove that $Pf(t, x^*(t)) = 0$ by the condition (36). Since $x^*(-t) = x^*(t)$ and $f(t, x^*)$ is a ω-periodic odd function with respect to t, we have

$$\begin{aligned} Pf(t, x^*(t)) &= \frac{1}{\omega} \int_0^\omega f(t, x^*(t)) \, dt \\ &= \frac{1}{\omega} \int_0^\omega f(t, x^*(-t)) \, dt \\ &= -\frac{1}{\omega} \int_0^\omega f(-t, x^*(-t)) \, dt \\ &= -\frac{1}{\omega} \int_0^\omega f(-t + \omega, x^*(-t + \omega)) \, dt \\ &\overset{\tau = -t + \omega}{=} -\frac{1}{\omega} \int_0^\omega f(\tau, x^*(\tau)) \, d\tau \end{aligned}$$

which implies $Pf(t, x^*(t)) = 0$. This completes the proof of Proposition 1.

Proof of Proposition 2

In fact, let $p(t, x) = \varepsilon\phi(t)x_1$, then the IVP can be decomposed into the following form:

$$\begin{cases} \dot{x}_1 = \varepsilon\phi(t)x_1 \\ \dot{x}_2 = \varepsilon B_0(x_1, x_2)\phi(t)x_1 \\ x_1(t_0) = x_{10}, \qquad x_2(t_0) = x_{20} \end{cases}. \tag{43}$$

Obviously, the solution of the first IVP with respect to x_1 is given by $x_1 = x_1(t, t_0, x_{10}) = Q(t, t_0)x_{10}$. The corresponding IVP with respect to x_2 in (43) is thus reduced to (9) in this case.

Under (9), one has

$$\begin{aligned} Q(-t, t_0) &= \exp\left\{\varepsilon \int_{t_0}^{-t} \phi(\tau) d\tau\right\} \\ &\overset{\sigma = -\tau}{=} \exp\left\{-\varepsilon \int_{-t_0}^{t} \phi(-\sigma) d\sigma\right\} \\ &= \exp\left\{\varepsilon \int_{-t_0}^{t} \phi(\sigma) d\sigma\right\} \\ &= \exp\left\{\varepsilon \int_{-t_0}^{t_0} \phi(\sigma) d\sigma\right\} Q(t, t_0) \\ &= \exp\left\{\varepsilon \int_0^{t_0} \phi(\sigma) d\sigma + \varepsilon \int_{-t_0}^{0} \phi(\sigma) d\sigma\right\} Q(t, t_0) \\ &\overset{\sigma = -\tau}{=} \exp\left\{\varepsilon \int_0^{t_0} \phi(\sigma) d\sigma - \varepsilon \int_{t_0}^{0} \phi(-\tau) d\tau\right\} Q(t, t_0) \\ &= Q(t, t_0). \end{aligned}$$

That is, the solution $x_1 = x_1(t, t_0, x_{10})$ is an even function. Furthermore, we also have

$$\begin{aligned} &Q(t+\omega, t_0) - Q(t, t_0) \\ &= Q(t, t_0)\left[\exp\left\{\varepsilon \int_t^{t+\omega} \phi(\tau) d\tau\right\} - I\right] \\ &\overset{\tau=\sigma+\omega}{=} Q(t, t_0)\left[\exp\left\{\varepsilon \int_{t-\omega}^{t} \phi(\sigma) d\sigma\right\} - I\right] \\ &= Q(t, t_0)\left[\exp\left\{-\varepsilon \int_{-t+\omega}^{-t} \phi(-\tau) d\tau\right\} - I\right] \end{aligned}$$

$$\begin{aligned} &Q(t, t_0)\left[\exp\left\{\varepsilon \int_{-t+\omega}^{-t} \phi(\tau) d\tau\right\} - I\right] \\ &= Q(t, t_0)\left[\exp\left\{-\varepsilon \int_{-t}^{-t+\omega} \phi(\tau) d\tau\right\} - I\right] \\ &\overset{\tau=-\sigma}{=} Q(t, t_0)\left[\exp\left\{\varepsilon \int_{t}^{t+\omega} \phi(-\sigma) d\sigma\right\} - I\right] \\ &= Q(t, t_0)\left[\exp\left\{-\varepsilon \int_{t}^{t+\omega} \phi(\sigma) d\sigma\right\} - I\right] \\ &= -\left[Q(t+\omega, t_0) - Q(t, t_0)\right]\exp\left\{-\varepsilon \int_{t}^{t+\omega} \phi(\sigma) d\sigma\right\}. \end{aligned}$$

It implies that $Q(t+\omega, t_0) = Q(t, t_0)$, i.e., the solution $x_1 = x_1(t, t_0, x_{10})$ is ω-periodic. Let the right-hand side of (9) be denoted by $q(t, x_2)$, then $q(t+\omega, x_2) = q(t, x_2)$ and $q(-t, x_2) = -q(t, x_2)$ for all $t \in R$. Therefore, the solution of (9) is also ω-periodic by Proposition 1. This completes the proof of the proposition.

Proof of Proposition 3

In fact, under the conditions, there exists a periodic function $z(t, x)$ such that (4)–(6) are satisfied. Let the switching function $S(t, x)$ be given by (11). Then, we have

$$\begin{aligned} \frac{\partial S}{\partial t} &= B^T\left[\left(\frac{\partial^2 z}{\partial t \partial x}\right)^T z + \left(\frac{\partial z}{\partial x}\right)^T \frac{\partial z}{\partial t}\right] \\ &= -B^T\left\{\left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x} B p\right)\right]^T z + \left(\frac{\partial z}{\partial x}\right)^T \frac{\partial z}{\partial x} B p\right\} \\ &= -B^T\left\{\sum_{j=1}^{m}\left(b_j \frac{\partial p_j}{\partial x} + p_j \frac{\partial b_j}{\partial x}\right)^T \left(\frac{\partial z}{\partial x}\right)^T z \right. \\ &\qquad \left. + \left(z^T \frac{\partial^2 z}{\partial x_l \partial x_k}\right) B p + \left(\frac{\partial z}{\partial x}\right)^T \frac{\partial z}{\partial x} B p\right\} \end{aligned} \tag{44}$$

$$\begin{aligned} \frac{\partial S}{\partial x} B &= B^T \left(\frac{\partial z}{\partial x}\right)^T \frac{\partial z}{\partial x} B \\ &\quad + B^T\left(z^T \frac{\partial^2 z}{\partial x_l \partial x_k}\right) B + \left(z^T \frac{\partial z}{\partial x} \frac{\partial b_i}{\partial x} b_j\right) \end{aligned} \tag{45}$$

where partial derivative symbol is used to avoid confusion. Hence, $(\partial S/\partial x)B$ is at least locally nonsingular since $(\partial z/\partial x)$ and B are full rank. Besides, from (44) and (45) one immediately obtains the following relation:

$$\frac{\partial S}{\partial t} + \frac{\partial S}{\partial x} B p = \left(z^T \frac{\partial z}{\partial x}[b_j, b_i]\right) p - B^T \left(\frac{\partial p}{\partial x}\right)^T B^T \left(\frac{\partial z}{\partial x}\right)^T z. \tag{46}$$

The sliding mode equation is thus given as follows:

$$\begin{cases} S = B^T \left(\frac{\partial z}{\partial x}\right)^T z = 0 \\ \dot{x} = B p + B\left(\frac{\partial S}{\partial x} B\right)^{-1} \left(z^T \frac{\partial z}{\partial x}[b_i, b_j]\right) p \end{cases}. \tag{47}$$

Let the Lyapunov function $V(t, x)$ be defined in (6), then $V(t, x)$ is periodic as indicated in [16] and

$$\dot{V}(t, x) = 0 \tag{48}$$

which implies that the sliding state satisfies $V(t, x) = \text{constant}$.

For any solution of (33) starting from the set

$$M = \left\{(t, x) : \dot{V}(t, x) = 0\right\} = \{(t, x) : S = 0\} \tag{49}$$

which remains also in M for all $t \geq t_r$ (t_r denotes the initial time when the sliding motion occurs) must be $x = 0$. In fact, since $z(t, x) = \text{constant}$, if $S = 0$, differentiating $z(t, x)$ and using both (44) and (47) we then obtain

$$\frac{\partial z}{\partial x} B \left(z^T \frac{\partial z}{\partial x}[b_i, b_j]\right) p = 0.$$

Since $(\partial z/\partial x)B$ is full rank, it also implies

$$\left(z^T \frac{\partial z}{\partial x}[b_i, b_j]\right) p = 0.$$

Using Assumption A2) and similar method to [16], further differentiation of $S = 0$ will yield the higher order Lie brackets. Since $p(t, x)$ only vanishes for $x = 0$, therefore, the set M does not contain a complete positive integral curve. The solution $x = 0$ is thus uniformly asymptotically stable by the Barbashin–Krasovskij theorem in [5].

References

[1] A. Astolfi, "Discontinuous control of nonholonomic systems," *Syst. Control Lett.*, vol. 27, no. 1, pp. 37–45, 1996.

[2] A. M. Bloch, M. Reyhanoglu, and N. H. McClamroch, "Control and stabilization of nonholonomic dynamic systems," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 1746–1757, Oct. 1992.

[3] A. M. Bloch and S. Drakunov, "Stabilization and tracking in the nonholonomic integrator via sliding modes," *Syst. Control Lett.*, vol. 29, no. 2, pp. 91–99, 1996.

[4] C. C.C. Canudas de Wit, B. Siciliano, and G. Bastin, Eds., *Theory of Robot Control*. London, U.K.: Springer-Verlag, 1998.

[5] M. Farkas, *Periodic Motions*. New York: Springer-Verlag, 1994.

[6] S. S. Ge, Z. Sun, T. H. Lee, and M. W. Spong, "Linearization and stabilization of second-order nonholonomic chained systems," *Int. J. Control*, vol. 74, no. 14, pp. 1383–1392, 2001.

[7] J. Guldner and V. I. Utkin, "Stabilization of nonholonomic mobile robots using Lyapunov functions for navigation and sliding mode control," in *Proc. 33rd IEEE Conf. Decision Control*, 1994, pp. 2967–2972.

[8] P. Hartman, *Ordinary Differential Equations*. Boston, MA: Birkhäuser, 1982.

[9] Z. P. Jiang, "Robust exponential regulation of nonholonomic systems with uncertainties," *Automatica*, vol. 36, pp. 189–209, 2000.

[10] Z. P. Jiang and H. Nijmeijer, "A recursive technique for tracking control of nonholonomic systems in chained form," *IEEE Trans. Automat. Contr.*, vol. 44, pp. 265–279, Feb. 1999.

[11] I. Kolmanovsky and N. H. McClamroch, "Developments in nonholonomic control problems," *IEEE Control Syst. Mag.*, vol. 15, pp. 20–36, June 1995.

[12] J. P. Laumond, Ed., *Robot Motion Planning and Control*. London, U.K.: Springer-Verlag, 1998.

[13] W. Lin, R. Pongvuthithum, and C. Qian, "Control of high-order nonholonomic systems in power chained form using discontinuous feedback," *IEEE Trans. Automat. Contr.*, vol. 47, pp. 108–115, Jan. 2002.

[14] R. M. Murry and S. S. Sastry, "Nonholonomic motion planning: Steering using sinusoids," *IEEE Trans. Automat. Contr.*, vol. 38, pp. 700–716, May 1993.
[15] G. Oriolo, S. Panzieri, and G. Ulivi, "An iterative learning controller for nonholonomic mobile robots," *Int. J. Robot. Res.*, vol. 17, no. 9, pp. 954–970, 1998.
[16] J. B. Pomet, "Explicit design of time-varying stabilizing control laws for a class of controllable systems without drift," *Syst. Control Lett.*, vol. 18, pp. 147–158, 1992.
[17] Z. Retchkiman and F. Khorrami, "The time optimal control problem for a class of nonholonomic systems: A case study," in *Proc. 13th Triennial World Congr.*, San Francisco, CA, 1996, pp. 449–453.
[18] C. Samson, "Control of chained systems application to path following and time-varying point-stabilization of mobile robots," *IEEE Trans. Automat. Contr.*, vol. 40, pp. 64–77, Jan. 1995.
[19] E. D. Sontag, "Control of systems without drift via generic loops," *IEEE Trans. Automat. Contr.*, vol. 40, pp. 1210–1219, July 1995.
[20] O. J. Sordalen and O. Egeland, "Exponential stabilization of nonholonomic chained systems," *IEEE Trans. Automat. Contr.*, vol. 40, pp. 35–49, Jan. 1995.
[21] C. Y. Su and Y. Stepanenko, "Robust motion/force control of mechanical systems with classical nonholonomic constraints," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 609–614, May 1994.
[22] V. I. Utkin, *Sliding Modes in Control and Optimizations*. New York: Springer-Verlag, 1992.

New Limit Power Function Spaces

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***Abstract*—To answer an open question, we propose two function spaces: One is Banach and another is Hilbert. It is shown that the Hilbert space is the largest one among those Hilbert spaces in limit power function set whose members have associated Fourier series (in sense of a new basis) and satisfy Parseval's equality.**

***Index Terms*—Almost-periodic functions, Fourier series, limit power functions, mean.**

I. Introduction

As in [14], a function f is called limit power if the limit

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(t)|^2\,dt$$

exists. Denote by H_2 the set of all such functions.

One important subset of H_2 is $\mathcal{AP}(\mathbb{R})$, the space of almost periodic functions (e.g., [4], [5], [8], [10], and [17]) and so is the Besicovitch space B_2 [3], the completion of $\mathcal{AP}(\mathbb{R})$ in H_2.

An example in [13] shows that H_2 is not closed under addition. The lack of closedness under addition caused some difficulties in such areas as Robust Control (e.g., see [11]). [13] opens the question: except for some subsets of H_2 which are already known to be vector spaces (e.g., $L_2((\mathbb{R}))$, $\{f\in L_\infty((\mathbb{R})) : \lim_{|t|\to\infty} f(t)\text{exists}\}$, $\mathcal{AP}(\mathbb{R})$), it is not clear whether a "nice" (e.g., Hilbert), large vector space could be defined. The background of [13] and related problems being pointed out by some authors (e.g., [11], [12], [14], and the references therein) show that new, larger, nice spaces are needed. The purpose of the paper is to define such spaces. Thus, we answer the open question affirmatively.

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II. Uniform Limit Power Functions

By direct calculation, one can show that

$$\lim_{T\to\infty}\frac{1}{T}\int_{0}^{T}(e^{i\lambda t^\alpha}\cdot e^{-i\mu t^\beta})\,dt=\begin{cases}1, & \alpha=\beta \quad \mu=\lambda\\ 0, & \text{otherwise}\end{cases}$$

where α, β are positive numbers and $\lambda, \mu\in\mathbb{R}$. That is, the set $\{e^{i\lambda t^\alpha}\}$ is orthonormal.

When $\alpha>1$, the functions $e^{i\lambda t^\alpha}$ have important applications. We refer the readers to [1], [2], [7], [9], [15], and [16] for details.

Remark 2.1: Since the domain of the function $e^{i\lambda t^\alpha}$ in general is $\mathbb{R}^+$, we consider $\mathbb{R}^+$ only in the paper. For special numbers of α, for example, α's are positive integers, the domain will be $\mathbb{R}$. In this case, all the results will hold for $\mathbb{R}$. For example, the limit will be

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}(e^{i\lambda t^\alpha}\cdot e^{-i\mu t^\beta})\,dt=\begin{cases}1, & \alpha=\beta \quad \mu=\lambda\\ 0, & \text{otherwise.}\end{cases}$$

In general, $\alpha>0$ one can also deal with $\mathbb{R}^-$ or $\mathbb{R}$ by using $|t|^\alpha$ rather than t^α.

We call the functions

$$\sum_{k=1}^{n}a_k e^{i\lambda_k t^\alpha}$$

α-trigonometric polynomials. As these functions are in $\mathcal{AP}(\mathbb{R})$, we have the following.

Definition 2.2: Let $\alpha>0$ be fixed. A function on $\mathbb{R}^+$ is said to have uniform limit power if for each $\epsilon>0$ there exists an α-trigonometric polynomial P_ϵ such that

$$\|f-P_\epsilon\|=\sup\{|f(t)-P_\epsilon(t)| : t\in(\mathbb{R}^+)\}<\epsilon. \tag{2.1}$$

Denote by $\mathcal{ULP}_\alpha(\mathbb{R}^+)$, the space of all such functions.

One sees that $\mathcal{ULP}_\alpha(\mathbb{R}^+)=\mathcal{AP}(\mathbb{R}^+)$ when $\alpha=1$.

It follows from Definition 2.2 that $\mathcal{ULP}_\alpha(\mathbb{R}^+)$ is the completion of α-trigonometric polynomials in $\mathcal{C}(\mathbb{R}^+)$, the space of bounded, continuous, complex-valued functions on $(\mathbb{R}^+)$ with supremum norm. Since the set of α-trigonometric polynomials is closed under addition, multiplication and conjugation, so is the completion $\mathcal{ULP}_\alpha(\mathbb{R}^+)$. Thus, we have shown the following result.

Theorem 2.3: $\mathcal{ULP}_\alpha(\mathbb{R}^+)$ is a C^*-subalgebra of $\mathcal{C}(\mathbb{R}^+)$ containing the constant functions.

We point out that there is a Fourier analysis theory for $\mathcal{ULP}_\alpha(\mathbb{R}^+)$ analogical to that of $\mathcal{AP}(\mathbb{R}^+)$. That is, for each $f\in\mathcal{ULP}_\alpha(\mathbb{R}^+)$ there exists a unique Fourier series $\sum_{n=1}^{\infty}A_n e^{i\lambda_n t^\alpha}$ and Parseval's equality $\lim_{T\to\infty}(1/T)\int_0^T|f(t)|^2dt=\sum_{n=1}^{\infty}|A_n|^2$ holds. To show this, we only need to set up some correspondence between $\mathcal{ULP}_\alpha(\mathbb{R}^+)$ and $\mathcal{AP}(\mathbb{R}^+)$. For the α-trigonometric polynomial P_ϵ in Definition 2.2, let $s=t^\alpha$. Then, P_ϵ becomes trigonometric polynomial of s. That is

$$P_\epsilon(s)=\sum_{k=1}^{n}a_k e^{i\lambda_k s}.$$