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Robust output tracking control for the systems with uncertainties

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A new robust tracking control is proposed for the minimum phase dynamical systems with parameter uncertainties and unmatched disturbance, where only the input–output measurement of the system is employed. The system parameters may vary slightly around their corresponding nominal values. The disturbance is assumed to be bounded. However, the upper and lower bounds are unknown. First, the frame of the control law is presented. Then, a special bounded signal is generated by the disturbance and the model uncertainties are estimated by a new non-linear method, where the upper and lower bounds of the special signal are adaptively updated online. Finally, the robust tracking control is synthesized by using the estimate of the special signal. The output tracking error can be made as small as necessary by choosing the design parameters. The attraction of the proposed method lies in its robustness to uncertainties and its ease of implementation. Example and simulation results are presented to show the effectiveness of the proposed algorithm.

1. Introduction

Robust output tracking control for minimum-phase dynamical systems has been studied for many years. Various robust design methodologies, which are usually based on the Lyapunov's direct method, have been proposed in the robust control literature, such as by Narendra and Annaswamy (1989), Sastry and Bodson (1989) and Slotine (1991). In recent years, the robust output tracking control for uncertain dynamical systems has been a topic of considerable interest since all the practical control systems are subjected to uncertainties. Asymptotic stability for output tracking of linear systems with constant disturbance has been shown in Schmitendorf and Barmish (1986, 1987), where the reference signal is composed of those functions generated from the unit step function of time by either integration or differentiation.

For the systems with much more general uncertainties, one outstanding approach is developed by Qu and Dawson (1992) and Zhu *et al.* (1995), where the input–output information and the *a priori* knowledge of the upper and lower bounds of the uncertainties are employed. The formulation of Qu and Dawson is based on the transfer function method, while the formulation of Zhu *et al.* is based on the state space technique. For the minimum-phase dynamical systems with relative degree one, the overall systems can be guaranteed to be exponentially stable. While for the systems with higher relative degrees, the overall systems can be ensured to be globally uniformly ultimately bounded (GUUB), which can be made arbitrarily close to exponential stability if the control energy permits.

Another typical approach for the systems with uncertainties is to design a VSS-type switching controller to stabilize the overall system. Only the output information and *a priori* knowledge of the upper bound of the disturbances are required in this kind of controller (Walcott and Zak 1988, El-Khazali and DeCarlo 1992, Spurgeon and Davies 1993, Zak and Hui 1993, Edwards and Spurgeon 1995, 1996). These proposed formulations are restricted to the minimum phase multi-input multi-output dynamical systems with relative degree one.

To deal with the systems with unmatched disturbances, a typical method (Rundell *et al.* 1996) based on the sliding mode techniques is proposed by using the state space approach. By estimating the states and at least some of their derivatives from the input–output information, the unmatched disturbances are estimated.

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The estimated disturbances and their derivatives are then applied to the robust controller formulation. It should be pointed out that the disturbance observer formulation results in a very high order structure, which is very complicated.

Corresponding to the above state space method, Chen *et al.* (1997) proposed a new method based strictly on the transfer function method in the frequency domain, where the input–output information and *a priori* knowledge of the upper and lower bounds of the disturbance were employed. The disturbance is composed of the model uncertainties, non-linearities, etc. First, a new formulation, which can be regarded as a kind of 'non-linear differentiator', is proposed to estimate the disturbance based on the VSS equivalent control method. Then, by using the estimate of a special signal generated by the disturbance, the intermediate control, which is a filter of the real control input, is determined to cancel the influence generated by the disturbance. Finally, the real control input is synthesized by using the proposed non-linear differentiator. However, the *a priori* knowledge of the upper and lower bounds of the disturbance may not be easily obtained in practice. Further, the VSS equivalent control method is not strict because, on the sliding surface $S(t) = 0$, it is not proved theoretically that the derivative of $S(t)$ is also zero.

This paper tries to consider the output tracking control for the minimum-phase dynamical systems with parameter uncertainties and unmatched disturbance. The system parameters may vary slightly around their corresponding nominal values. The disturbance is assumed to be bounded. However, the bound is unknown. A special signal generated by the disturbance and model uncertainties is estimated online by our new non-linear method without using the equivalent control method, where the bound of the special signal is adaptively updated. The almost perfect output tracking control is formulated by using the online estimate of the special signal to cancel the influence generated by the disturbance and the model uncertainties. The uniform boundedness of all the signals in the loop is guaranteed. The tracking error can be made as small as necessary by choosing the design parameters. The attraction of the proposed controller lies in the robustness to uncertainties and the ease of implementation.

This paper is organized as follows. Section 2 gives the problem formulation. In Section 3, the frame of the output-tracking controller is constructed. In Section 4, a special signal generated by the disturbance and model uncertainties is estimated. Section 5 constructs the concrete robust control law and analyses the stability of the closed-loop system. In Section 6, a design example and simulations are presented to show the effectiveness of the proposed algorithm. Section 7 concludes the paper.

2. Problem statement

Consider an uncertain system of the form

$$a'(s)y(t) = b'(s)u(t) + v(t), \tag{1}$$

where s is the differential operator; $u(t)$ and $y(t)$ are the scalar input and output, respectively; $v(t)$ is an unknown signal composed of model uncertainties, non-linearities and disturbances, etc.; and where $a'(s)$ and $b'(s)$ are described by

$$a'(s) = s^n + a'_1 s^{n-1} + \cdots + a'_{n-1}s + a'_n, \tag{2}$$

$$b'(s) = b_r s^{n-r} + b'_{r-1}s^{n-r+1} + \cdots + b'_{n-1}s + b'_n. \tag{3}$$

For simplicity, the signal $v(t)$ is called the 'disturbance' of the system in the present paper.

We make the following assumptions:

Assumption A1: *$a'(s)$ and $b'(s)$ are coprime; $b'(s)$ is a Hurwitz polynomial.*

Assumption A2: *Indices n and r are known; further, it is assumed that the parameter $b_r (b_r \neq 0)$ is known.*

For the polynomials $a'(s)$ and $b'(s)$, they are expressed as

$$\begin{aligned} a'(s) &= (s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n) \\ &\quad + (\Delta a_1 s^{n-1} + \cdots + \Delta a_{n-1}s + \Delta a_n) \\ &\triangleq a(s) + \Delta a(s), \end{aligned} \tag{4}$$

$$\begin{aligned} b'(s) &= (b_r, s^{n-r} + b_{r-1}s^{n-r+1}_{\cdots} + b_{n-1}s + b_n) \\ &\quad + (\Delta b_{r-1}s^{n-r+1} + \cdots + \Delta b_{n-1}s + \Delta b_n) \\ &\triangleq b(s) + \Delta b(s). \end{aligned} \tag{5}$$

If the parameters in $\Delta a(s)$ and $\Delta b(s)$ are very small, it can be seen that $b(s)$ is still a Hurwitz polynomial, and $a(s)$ and $b(s)$ are coprime. Another assumption on the polynomials $a'(s)$ and $b'(s)$ is made as follows.

Assumption A3: *Parameters in $a(s)$ and $b(s)$ are known, the parameters in $\Delta a(s)$ and $\Delta b(s)$ are very small, and $b(s)$ is a Hurwitz polynomial.*

Now, rewrite (1) as

$$a(s)y(t) = b(s)(u(t) + \bar{v}(t)), \tag{6}$$

where $\bar{v}(t)$ is defined as

$$\bar{v}(t) = -\frac{\Delta a(s)}{b(s)}y(t) + \frac{\Delta b(s)}{b(s)}u(t) + \frac{1}{b(s)}v(t). \tag{7}$$

The control purpose is to drive the output to track a desired uniformly bounded signal $y_d(t)$ for the uncertain systems (1) subjected to the assumptions (**A1–3**), where $y_d(t)$ is differentiable to a necessary order and the derivatives are also uniformly bounded. The controller will be designed based on equation (6).

Remark 1: For simplicity, the parameter b_r is assumed known (Sastry and Bodson 1989). Further, it should be noted that $\Delta b(s)/b(s)$ is a strictly proper polynomial.

Remark 2: Many motion control systems can be described by equation (6), which is in the formulation of a nominal system plus an additional uncertain signal (Umeno *et al.* 1993, Casini *et al.* 1995, Chen *et al.* 2000, Komada *et al.* 2000). Thus, it is important to deal with the control problem for this class of uncertain systems even though it is assumed that $a(s)$ and $b(s)$ are known.

3. Frame of the output tracking control

It is worth mentioning that the discussions here, as well as those corresponding discussions in other sections, are based on the transfer function method, which inherently assumes zero initial conditions for all internal states of the system. Fortunately, this treatment does not lose any generality since, for a stable closed-loop linear system, non-zero initial conditions only contribute to the solution of the state (or the system output) an additive term which decays to zero exponentially. Thus, the initial conditions of the filtered signals can be assumed to be zero.

We now introduce monic Hurwitz polynomials $d(s)$ and $h(s)$ of orders n and r, respectively. Consider the following equation (Sastry and Bodson 1989):

$$d(s)h(s) = \eta(s)a(s) + \mu(s), \tag{8}$$

where $\eta(s)$ is a monic rth order polynomial, $\mu(s)$ is a $(n-1)$-th order polynomial. It is very clear that the solutions $\eta(s)$ and $\mu(s)$ exist uniquely.

Multiplying (8) by $y(t)$ and applying equation (6) yields:

$$d(s)h(s)y(t) = \eta(s)b(s)\{u(t) + \bar{v}(t)\} + \mu(s)y(t). \tag{9}$$

By adding and subtracting the term $b_r d(s)\{u(t) + \bar{v}(t)\}$ in the right-hand side of equation (9) and then dividing the both sides by $d(s)$, it gives

$$\begin{aligned} h(s)y(t) &= b_r\{u(t) + \bar{v}(t)\} \\ &\quad + \frac{\eta(s)b(s) - b_r d(s)}{d(s)}\{u(t) + \bar{v}(t)\} \\ &\quad + \frac{\mu(s)}{d(s)}y(t). \end{aligned} \tag{10}$$

Since $(\eta(s)b(s) - b_r d(s))/d(s)$ is strictly proper, it can be seen that $((\eta(s)b(s) - b_r d(s))/d(s))\ u(t)$ is a computable signal if the control input $u(t)$ is determined. The frame of the control input will be determined based on equation (10).

Here, the control input is considered in the following form

$$\begin{aligned} u(t) = -\omega(t) - \frac{1}{b_r}\bigg\{ &\frac{\eta(s)b(s) - b_r d(s)}{d(s)}\{u(t) + \omega(t)\} \\ &+ \frac{\mu(s)}{d(s)}y(t) - h(s)y_d(t)\bigg\}, \end{aligned} \tag{11}$$

where $\omega(t)$ is a bounded signal which will be generated in the next section.

By multiplying the both sides of (11) with $b_r d(s)$ and rearranging it, it yields

$$\mu(s)y(t) + \eta(s)b(s)u(t) = -\eta(s)b(s)\omega(t) + d(s)h(s)y_d(t). \tag{12}$$

Thus, from (1) and (12), the closed-loop system can be expressed as

$$\begin{aligned} \begin{bmatrix} a'(s) & -b'(s) \\ \mu(s) & \eta(s)b(s) \end{bmatrix} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} v(t) - \begin{bmatrix} 0 \\ \eta(s)b(s) \end{bmatrix} \omega(t) \\ &\quad + \begin{bmatrix} 0 \\ d(s)h(s) \end{bmatrix} y_d(t). \end{aligned} \tag{13}$$

Since

$$\begin{aligned} \det \begin{bmatrix} a'(s) & -b'(s) \\ \mu(s) & \eta(s)b(s) \end{bmatrix} &= d(s)h(s)b(s) + \eta(s)b(s)\Delta a(s) \\ &\quad + \mu(s)\Delta b(s) \\ &\triangleq \Omega(s), \end{aligned} \tag{14}$$

it can be seen that $\Omega(s)$ is also a Hurwitz polynomial if the parameters in $\Delta a(s)$ and $\Delta b(s)$ are very small (see **A3**). Thus, from (13), it can be concluded that if the disturbance $v(t)$ is bounded, all the signals in the loop remain bounded by observing that $\omega(t)$ and $y_d(t)$ are bounded signals.

So, it is reasonable to make the following assumption about $v(t)$.

Assumption A4: *The disturbance $v(t)$ is bounded. However, its upper and lower bounds are unknown. Further, if $r = n$, it is also assumed that the derivative of $v(t)$ is bounded.*

Now, by combining (6) and (12), the closed-loop system can also be expressed as

$$\begin{aligned} \begin{bmatrix} a(s) & -b(s) \\ \mu(s) & \eta(s)b(s) \end{bmatrix} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} &= \begin{bmatrix} b(s) \\ 0 \end{bmatrix} \bar{v}(t) - \begin{bmatrix} 0 \\ \eta(s)b(s) \end{bmatrix} \omega(t) \\ &\quad + \begin{bmatrix} 0 \\ d(s)h(s) \end{bmatrix} y_d(t). \end{aligned} \tag{15}$$

By pre-multiplying the both sides of (15) with

$$\operatorname{adj} \begin{bmatrix} a(s) & -b(s) \\ \mu(s) & \eta(s)b(s) \end{bmatrix},$$

it yields

$$d(s)h(s)b(s)\begin{bmatrix} y(t) \\ u(t) \end{bmatrix} = b(s)\begin{bmatrix} \eta(s)b(s) \\ -\mu(s) \end{bmatrix}\bar{v}(t) - \eta(s)b(s)\begin{bmatrix} b(s) \\ a(s) \end{bmatrix}\omega(t) + d(s)h(s)\begin{bmatrix} b(s) \\ a(s) \end{bmatrix} y_d(t). \quad (16)$$

For the output $y(t)$, from (16), it gives

$$d(s)h(s)(y(t) - y_d(t)) = \eta(s)b(s)(\bar{v}(t) - \omega(t)). \quad (17)$$

From (17), it can be argued that if $\omega(t)$ is very close to $\bar{v}(t)$, the output tracking error can be controlled very small. Thus, the remaining task is to estimate the signal $\bar{v}(t)$.

4. Estimation of the signal $\bar{v}(t)$

Here, $(1/(s+\lambda)^{r-1})\bar{v}(t)$ is estimated in the first step. In the second step, $(1/(s+\lambda)^{r-2})\bar{v}(t)$ is estimated by using the estimate of $(1/(s+\lambda)^{r-1})\bar{v}(t)$. In the ith step, $(1/(s+\lambda)^{r-i})\bar{v}(t)$ is estimated by using the estimates of $(1/(s+\lambda)^{r-j})\bar{v}(t)$ $(j = 1, \ldots, i-1)$. Finally, in the rth step, the signal $\bar{v}(t)$ is estimated by using the estimates of $(1/(s+\lambda)^{r-j})\bar{v}(t)$ $(j = 1, \ldots, r-1)$.

First, the boundedness of the signal $\bar{v}(t)$ is clarified by the next lemma.

Lemma 1: *The signal $\bar{v}(t)$ and its first-order derivative are bounded in the closed-loop system.*

Proof: See appendix A.

By Lemma 1, for any $i \geq 0$, it is easy to see that the filtered signals $(1/(s+\lambda)^i)\bar{v}(t)$ are also bounded, i.e. there exist finite positive constants C_i such that

$$\left| \frac{1}{(s+\lambda)^i}\bar{v}(t) \right| \leq C_i \quad (18)$$

for $t \geq t_0$, where t_0 is the starting instant, λ is a positive constant, C_i are unknown constants. In the proposed formulation, the upper bounds C_i are also adaptively updated online.

Motivated by the formulation of the state observer for a possibly unstable system (Kreisselmeier 1977), we introduce a monic nth order Hurwitz polynomial:

$$f(s) = \frac{1}{b_r} b(s)(s+\lambda)^r. \quad (19)$$

Now, rewriting equation (6) as

$$y(t) = \frac{f(s) - a(s)}{f(s)} y(t) + \frac{b_r}{(s+\lambda)^r} u(t) + \frac{b_r}{(s+\lambda)^r}\bar{v}(t). \quad (20)$$

Multiplying both sides of equation (20) with $(s+\lambda)$ yields

$$\dot{y}(t) + \lambda y(t) = b_r \frac{f(s) - a(s)}{b(s)(s+\lambda)^{r-1}} y(t) + \frac{b_r}{(s+\lambda)^{r-1}} u(t) + \frac{b_r}{(s+\lambda)^{r-1}}\bar{v}(t). \quad (21)$$

It is easy to know that

$$b_r((f(s) - a(s))/(b(s)(s+\lambda)^{r-1}))y(t)$$

is a signal that can be calculated.

The proposed formulation can be summarized in the following.

Step 1. Based on equation (21), $(1/(s+\lambda)^{r-1})\bar{v}(t)$ is estimated.

Motivated by the formulation of the Luenberger-type state observer, consider the dynamical system described by

$$\dot{\hat{y}}(t) + \lambda\hat{y}(t) = b_r \frac{f(s) - a(s)}{b(s)(s+\lambda)^{r-1}} y(t) + \frac{b_r}{(s+\lambda)^{r-1}} u(t) + b_r w_1(t), \ \hat{y}(t_0) = y(t_0), \quad (22)$$

where $w_1(t)$ is a signal to be determined, $\hat{y}(t)$ is the solution of the differential equation in (22).

Combining (21) and (22) yields

$$\dot{\bar{y}}(t) + \lambda\bar{y}(t) = b_r \left\{ \frac{1}{(s+\lambda)^{r-1}}\bar{v}(t) - w_1(t) \right\}, \quad (23)$$

where $\bar{y}(t) = y(t) - \hat{y}(t)$. It can be seen that, if the signal $w_1(t)$ can be formulated such that $\bar{y}(t)$ and $\dot{\bar{y}}(t)$ are very small, then $w_1(t)$ can be regarded as the approximate estimate of $(1/(s+\lambda)^{r-1})\bar{v}(t)$.

Remark 3: If the upper bound of $|(1/(s+\lambda)^{r-1})\bar{v}(t)|$ is known, then $w_1(t)$ can be easily determined such that $\bar{y}(t)$ is very small based on the variable structure control method (Utkin 1992). However, the problem is that the upper bound of $|(1/(s+\lambda)^{r-1})\bar{v}(t)|$ is unknown and $\dot{\bar{y}}(t)$ also needs to be very small.

In this research, the formulation of $w_1(t)$ is motivated by the variable structure method and the adaptive control technique. By adaptively updating the upper bound C_{r-1} of $|(1/(s+\lambda)^{r-1})\bar{v}(t)|$, the signal $w_1(t)$ is considered as

$$w_1(t) = \frac{b_r\{y(t) - \hat{y}(t)\}\hat{C}_{r-1}(t)}{|b_r\{y(t) - \hat{y}t\}| + \delta_1}, \tag{24}$$

where δ_1 is a small positive constant, $\hat{C}_{r-1}(t)$ is updated by the following adaptive algorithm

$$\dot{\hat{C}}_{r-1}(t) = \begin{cases} \alpha_{r-1}|y(t) - \hat{y}(t)| & \text{if } |y(t) - \hat{y}(t)| > \sqrt{\dfrac{2\delta_1 \hat{C}_{r-1}}{\lambda}}, \\ 0 & \text{otherwise} \end{cases} \tag{25}$$

$\hat{C}_{r-1}(t_0)$ can be chosen as any small positive constant, α_{r-1} is a positive constant.

It can be proved that $\bar{y}(t)$ and $\dot{\bar{y}}(t)$ are uniformly bounded and there exist $\gamma_{1i}(\delta_1) > 0$ and $T_1 > 0$ such that

$$|\bar{y}(t)| \leq \gamma_{11}(\delta_1) \tag{26}$$

$$|\dot{\bar{y}}(t)| \leq \gamma_{12}(\delta_1), \tag{27}$$

as $t > T_1$, where $\gamma_{1i}(\delta_1) \to 0$ as $\delta_1 \to 0$. The proofs of (26) and (27) are given in appendices B and C, respectively.

Therefore, by combining (23), (26) and (27), it is concluded that there exists $\varepsilon_1(\delta_1) > 0$ such that

$$\left| \frac{1}{(s+\lambda)^{r-1}} \bar{v}(t) - w_1(t) \right| \leq \varepsilon_1(\delta_1), \tag{28}$$

as $t > T_1$, where $\varepsilon_1(\delta_1) \to 0$ as $\delta_1 \to 0$. Thus, $w_1(t)$ is the approximate estimate of $(1/(s+\lambda)^{r-1})\bar{v}(t)$.

Step 2. $(1/(s+\lambda)^{r-2})\bar{v}(t)$ is estimated by using $w_1(t)$ based on the next trivial equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{(s+\lambda)^{r-1}} \bar{v}(t) \right) + \frac{\lambda}{(s+\lambda)^{r-1}} \bar{v}(t) = \frac{1}{(s+\lambda)^{r-2}} \bar{v}(t). \tag{29}$$

Corresponding to equation (29), the next differential equation is considered

$$\dot{\hat{w}}_1(t) + \lambda \hat{w}_1(t) = w_2(t), \; \hat{w}_1(t_0) = 0, \tag{30}$$

where $w_2(t)$ is a signal to be determined, $\hat{w}_1(t)$ is the solution of the differential equation in (30).

Denote $\bar{w}_1(t) = (1/(s+\lambda)^{r-1})\bar{v}(t) - \hat{w}_1(t)$. Then, from (29) and (30), it yields

$$\dot{\bar{w}}_1(t) + \lambda \bar{w}_1(t) = \frac{1}{(s+\lambda)^{r-2}} \bar{v}(t) - w_2(t). \tag{31}$$

By mimicking the formulation of the first step, $w_2(t)$ is chosen as

$$w_2(t) = \hat{C}_{r-2}(t) \frac{w_1(t) - \hat{w}_1(t)}{|w_1(t) - \hat{w}_1(t)| + \delta_2}, \tag{32}$$

where δ_2 is a small positive constant, $\hat{C}_{r-2}(t)$ is updated by the following adaptive algorithm

$$\dot{\hat{C}}_{r-2}(t) = \begin{cases} \alpha_{r-2}|w_1(t) - \hat{w}_1(t)| & \text{if } |w_1(t) - \hat{w}_1(t)| > \sqrt{\dfrac{2\delta_2 \hat{C}_{r-2}}{\lambda}}, \\ 0 & \text{otherwise} \end{cases} \tag{33}$$

$\hat{C}_{r-2}(t_0)$ can be chosen as any small positive constant, α_{r-2} is a positive constant.

Remark 4: Since $(1/(s+\lambda)^{r-1})\bar{v}(t)$ is not an available signal, its estimate $w_1(t)$ replaces it in the formulation of $w_2(t)$ in (32).

It can be proved that $\bar{w}_1(t)$ and $\dot{\bar{w}}_1(t)$ are uniformly bounded and

$$|\bar{w}_1(t)| \leq \gamma_{21}(\delta_1, \delta_2) \tag{34}$$

$$|\dot{\bar{w}}_1(t)| \leq \gamma_{22}(\delta_1, \delta_2), \tag{35}$$

where $\gamma_{2i}(\delta_1, \delta_2) \to 0$ as $\sum_{j=1}^{2} \delta_j \to 0$. The proof of (34) is given in appendix D. Relation (35) can be similarly proved by referring the proof of (27).

Therefore, combining (31), (34) and (35) yields that there exist $\varepsilon_2(\delta_1, \delta_2) > 0$ and $T_2 > 0$ such that

$$\left| \frac{1}{(s+\lambda)^{r-2}} \bar{v}(t) - w_2(t) \right| \leq \varepsilon_2(\delta_1, \delta_2), \tag{36}$$

as $t > T_2$, where $\varepsilon_2(\delta_1, \delta_2) \to 0$ as $\sum_{j=1}^{2} \delta_j \to 0$. Thus, $w_2(t)$ is the approximate estimate of $(1/(s+\lambda)^{r-2})\bar{v}(t)$.

Step 3. $\zeta (2 < \zeta < r)$. $(1/(s+\lambda)^{r-\zeta})\bar{v}(t)$ is estimated based on the next trivial equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{ \frac{1}{(s+\lambda)^{r-\zeta+1}} \bar{v}(t) \right\} + \frac{\lambda}{(s+\lambda)^{r-\zeta+1}} \bar{v}(t) = \frac{1}{(s+\lambda)^{r-\zeta}} \bar{v}(t). \tag{37}$$

Consider the following differential equation

$$\begin{aligned} &\dot{\hat{w}}_{\zeta-1}(t) + \lambda \hat{w}_{\zeta-1}(t) = w_\zeta(t), \\ &w_\zeta(t) = \hat{C}_{r-\zeta}(t) \frac{w_{\zeta-1}(t) - \hat{w}_{\zeta-1}(t)}{|w_{\zeta-1}(t) - \hat{w}_{\zeta-1}(t)| + \delta_\zeta}, \\ &\hat{w}_{\zeta-1}(t_0) = 0, \end{aligned} \tag{38}$$

where $\hat{w}_{\zeta-1}(t)$ is a signal which can be generated by solving the differential equation in (38), δ_ζ is a small positive constant, $\hat{C}_{r-\zeta}(t)$ is updated by the following adaptive algorithm

$$\dot{\hat{C}}_{r-\varsigma}(t) = \begin{cases} \alpha_{r-\varsigma}|w_{\varsigma-1}(t) - \hat{w}_{\varsigma-1}(t)| & \text{if } |w_{\varsigma-1}(t) - \hat{w}_{\varsigma-1}(t)| > \sqrt{\dfrac{2\delta_\varsigma \hat{C}_{r-\varsigma}}{\lambda}}, \\ 0 & \text{otherwise} \end{cases} \quad (39)$$

$\hat{C}_{r-\varsigma}(t_0)$ can be chosen as any small positive constant, $\alpha_{r-\varsigma}$ is a positive constant.

By applying the results in the above $(\zeta - 1)$ steps, it can be similarly proved that there exist $T_\varsigma > 0$ and $\varepsilon_\zeta(\delta_1, \ldots, \delta_\zeta) > 0$ such that

$$\left| \frac{1}{(s+\lambda)^{r-\zeta}} \bar{v}(t) - w_\zeta(t) \right| \le \varepsilon_\zeta(\delta_1, \ldots, \delta_\zeta), \quad (40)$$

as $t > T_\varsigma$, where $\varepsilon_\zeta(\delta_1, \ldots, \delta_\zeta) \to 0$ as $\sum_{j=1}^{\zeta} \delta_j \to 0$.

By mathematical induction method, we get the next theorem.

Theorem 1: *For small positive constants $\delta_i > 0$ $(i = 1, \ldots, r)$, construct the dynamical systems described by*

$$\dot{\hat{y}}(t) + \lambda \hat{y}(t) = b_r \frac{f(s) - a(s)}{b(s)(s+\lambda)^{r-1}} y(t) + \frac{b_r}{(s+\lambda)^{r-1}} u(t) + b_r w_1(t),$$
$$\hat{y}(t_0) = y(t_0) \quad (22)$$

$$\dot{\hat{w}}_{i-1}(t) + \lambda \hat{w}_{i-1}(t) = w_i(t), \;\; \hat{w}_{i-1}(t_0) = 0, \;\; (i = 2, \ldots, r), \quad (41)$$

where $w_i(t)$ $(i = 1, \ldots, r)$ are given as

$$w_1(t) = \frac{b_r\{y(t) - \hat{y}(t)\}\hat{C}_{r-1}(t)}{|b_r\{y(t) - \hat{y})(t)\}| + \delta_1} \quad (24)$$

and

$$w_i(t) = \frac{\{w_{i-1}(t) - \hat{w}_{i-1}(t)\}\hat{C}_{r-i}(t)}{|w_{i-1}(t) - \hat{w}_{i-1}(t)| + \delta_i}, \qquad (i = 2, \ldots, r) \quad (42)$$

respectively; $\hat{C}_{r-i}(t)$ is updated by the following adaptive algorithm

$$\dot{\hat{C}}_{r-1}(t) = \begin{cases} \alpha_{r-1}|y(t) - \hat{y}(t)| & \textit{if } |y(t) - \hat{y}(t)| > \sqrt{\dfrac{2\delta_1 \hat{C}_{r-1}}{\lambda}} \\ 0 & \textit{otherwise} \end{cases} \quad (25)$$

$$\dot{\hat{C}}_{r-i}(t) = \begin{cases} \alpha_{r-i}|w_{i-1}(t) - \hat{w}_{i-1}(t)| & \textit{if } |w_{i-1}(t) - \hat{w}_{i-1}(t)| > \sqrt{\dfrac{2\delta_i \hat{C}_{r-i}}{\lambda}}, \\ 0 & \textit{otherwise} \end{cases}$$
$$(i = 2, \ldots r), \quad (43)$$

where $\hat{C}_{r-i}(t_0)$ can be chosen as any small positive constants, and α_{r-i} are positive constants for $i = 1, \ldots, r$. It can be concluded that, when $\sum_{j=1}^{r} \delta_j$ is very small, $w_i(t)$ can be approximately regarded as the corresponding estimates of $(1/(s+\lambda)^{r-i})\bar{v}(t)$, i.e. there exist $\varepsilon_i(\delta_1, \ldots, \delta_i) > 0$ and $T_i > 0$ such that

$$\left| \frac{1}{(s+\lambda)^{r-i}} \bar{v}(t) - w_i(t) \right| \le \varepsilon_i(\delta_1, \ldots, \delta_i), \quad (44)$$

as $t > T_i$, where $\varepsilon_i(\delta_1, \ldots, \delta_i) \to 0$ as $\sum_{j=1}^{i} \delta_j \to 0$ for $i = 1, \ldots, r$. Particularly, $w_r(t)$ is the approximate estimate of $\bar{v}(t)$.

Remark 5: By appendix C, it can be seen that the boundedness of $\dot{\bar{v}}(t)$ is necessary (which means that the boundedness of $\dot{v}(t)$ is needed if $r = n$) in order to assure that $\psi_r(t)$ (on the analogy of $\psi_1(t)$ in the first step) is uniformly bounded in the last step.

Remark 6: The design parameter $\lambda > 0$ determines the estimating speed. The estimating precision is dominated by the parameters λ and δ_i $(i = 1, \ldots, r)$. The design parameters $\delta_i > 0$ $(i = 1, \ldots, r)$ should be chosen very small. The parameters $\alpha_{r-i} > 0$ $(i = 1, \ldots, r)$ should be chosen large enough to adjust the estimated upper bounds $\hat{C}_{r-i}(t)$ rapidly.

5. Output tracking control and the closed-loop analysis

By choosing the signal $\omega(t)$ as $w_r(t)$, the next theorem gives the stability of the control system.

Theorem 2. *If $u(t)$ is chosen as* (11), *where $\omega(t)$ is replaced by $w_r(t)$ generated in Theorem 1, then all the signals in the closed-loop system remain bounded. Further, there exist $T > t_0$ and $\varepsilon(t, \delta_1, \ldots, \delta_r) > 0$ such that*

$$|y(t) - y_d(t)| < \varepsilon(t, \delta_1, \ldots, \delta_r) \quad (45)$$

for all $t > T$, where $\varepsilon(t, \delta_1, \ldots \delta_r) \to 0$ as $t \to \infty$ and $\sum_{i=1}^{r} \delta_i \to 0$.

Proof: The uniform boundedness of all the signals in the closed-loop system can be easily obtained by observing (13) and the discussions in Section 3. Further, by the choice of $\omega(t)$, (17) becomes

$$d(s)h(s)(y(t) - y_d(t)) = \eta(s)b(s)(\bar{v}(t) - w_r(t)). \quad (46)$$

By applying Theorem 1, the result is obvious.

Remark 7: It can be seen that the output tracking error is eventually controlled by the parameters λ and δ_i $(i = 1, \ldots, r)$.

Remark 8: Even though $\bar{v}(t)$ in (7) is not bounded in the open-loop system, it is bounded in the closed-loop system. The proposed robust controller can cope with a wide-ranging class of uncertain systems, which may include parameter uncertainties, unmodelled dynamics, non-linearities, etc.

6. Example and simulation results

Consider the system described by

$$(s+2)(s-1)^2 y(t) = (s+1)u(t) + v(t), \ y(0) = 0, \quad (47)$$

where the disturbance $v(t)$ is governed by

$$v(t) = (s-1)\left\{(\sin\ t)\frac{\dot{y}(t) + 0.5u(t) + 0.6y(t)}{|\dot{y}(t) + 0.5u(t) + 0.6y(t)| + 1}\ y(t)\right\}. \quad (48)$$

The purpose of the control is to drive the output to follow the signal $y_d(t) = 2\cos t$.

Now, we rewrite the system (47) as

$$(s+2)(s-1)^2 y(t) = (s+1)(u(t) + \bar{v}(t)), \quad (49)$$

where

$$\bar{v}(t) = \frac{1}{s+1} v(t). \quad (50)$$

Choose the Hurwitz polynomial $f(s)$ in (19) as

$$f(s) = (s+1)(s+3)^2, \quad (51)$$

where λ is chosen as $\lambda = 3$. Corresponding to (21), we have

$$\dot{y}(t) + 3y(t) = \frac{7s^2 + 18s + 7}{(s+1)(s+3)} y(t) + \frac{1}{s+3} u(t) + \frac{1}{s+3}\bar{v}(t). \quad (52)$$

From Theorem 1, we construct the following dynamical systems:

$$\dot{\hat{y}}(t) + 3\hat{y}(t) = \frac{7s^2 + 18s + 7}{(s+1)(s+3)} y(t) + \frac{1}{s+3} u(t) + w_1(t),$$
$$\hat{y}(0) = y(0), \quad (53)$$

$$\dot{\hat{w}}_1(t) + 3\hat{w}_1(t) = w_2(t), \ \hat{w}_1(0) = 0, \quad (54)$$

where $w_1(t)$ and $w_2(t)$ are respectively chosen as

$$w_1(t) = \frac{\{y(t) - \hat{y}(t)\}\hat{C}_1(t)}{|y(t) - \hat{y}(t)| + \delta_1}, \quad (55)$$

$$w_2(t) = \frac{\{w_1(t) - \hat{w}_1(t)\}\hat{C}_0(t)}{|w_1(t) - \hat{w}_1(t)| + \delta_2}; \quad (56)$$

$\hat{C}_1(t)$ and $\hat{C}_0(t)$ are respectively determined by

$$\dot{\hat{C}}_1(t) = \begin{cases} \alpha_1 |y(t) - \hat{y}(t)| & \text{if } |y(t) - \hat{y}(t)| > \sqrt{\frac{2}{3}\delta_1 \hat{C}_1(t)}, \ \hat{C}_1(0) = 0.5 \\ 0 & \text{otherwise} \end{cases} \quad (57)$$

$$\dot{\hat{C}}_0(t) = \begin{cases} \alpha_0 |w_1(t) - \hat{w}_1(t)| & \text{if } |w_1(t) - \hat{w}_1(t)| > \sqrt{\frac{2}{3}\delta_2 \hat{C}_0(t)}, \ \hat{C}_0(0) = 0.5. \\ 0 & \text{otherwise} \end{cases} \quad (58)$$

Therefore, $w_2(t)$ can be regarded as the approximate estimate of the disturbance $\bar{v}(t)$.

Choose the polynomials $h(s)$ and $d(s)$ in (8) as

$$h(s) = (s+2)(s+3), \ d(s) = (s+3)^3(s+1). \quad (59)$$

Solving equation (8) yields

$$\eta(s) = s^2 + 12s + 59, \quad (60)$$

$$\mu(s) = 160(s - 0.2)(s+2). \quad (61)$$

Therefore, the control should be chosen as

$$u(t) = -w_2(t) - \left\{\frac{6s+50}{(s+3)^2}\{u(t) + w_2(t)\} + \frac{160(s-0.2)(s+2)}{(s+1)(s+3)^2} y(t) - 2(s+2)(s+3)\ \cos t\right\}. \quad (62)$$

In the simulation process, the sampling period is chosen as 1×10^{-4} seconds. The parameters are chosen as $\delta_1 = \delta_2 = 2 \times 10^{-4}$, $\alpha_1 = \alpha_2 = 10$. The starting time is $t_0 = 0$. Figure 1 shows the difference between the signal $\bar{v}(t)$ and its estimate $w_2(t)$. It can be seen that a very good estimation is obtained. Figure 2 shows the output tracking control input. It can be seen the control input remains uniformly bounded. Figure 3 shows the difference between the controlled output and the desired output. It can be seen that the proposed control works very well. If the parameters δ_1 and δ_2 are chosen to be much smaller, the output tracking performance may become much better.

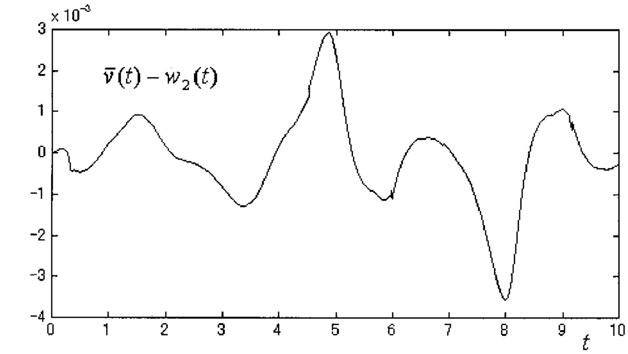

Figure 1. Difference between the signal $\bar{v}(t)$ and its estimate $w_2(t)$.

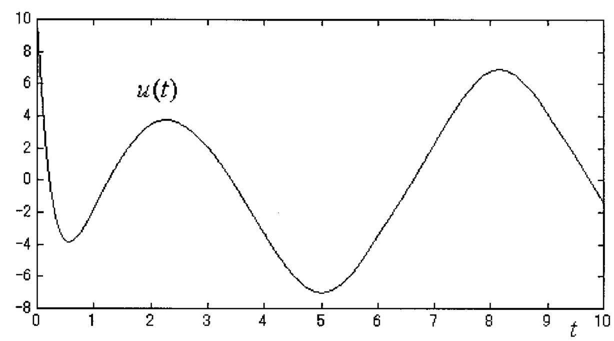

Figure 2. Robust output tracking control input.

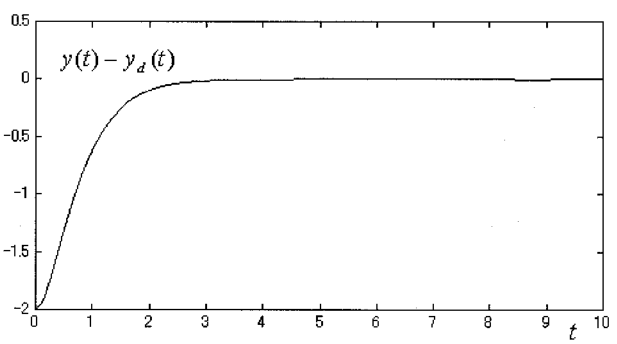

Figure 3. Difference between controlled system output and the desired output.

Remark 9: When the signals are implemented by digital computer, the design parameters δ_i should not be chosen to be $< 2T$, where T is the sampling period. Otherwise, as the variations of $w_1(t)$ will be too fast with respect to the sampling frequency, the differential equations in (22) and (41) cannot be precisely solved.

7. Conclusions

In this paper, a new robust output tracking controller is formulated for minimum-phase dynamical systems with parameter uncertainties and disturbances by using only the input–output information. The system parameters may vary slightly around their corresponding nominal values. The disturbance is assumed bounded but the bound is unknown. First, the control frame is given. Then, a special bounded signal generated by the disturbance and the model uncertainties are estimated, where the bound of the special signal is adaptively updated. Finally, by using the estimate of the special signal, the robust output tracking controller is formulated. All the signals in the closed-loop system are bounded, and the output tracking error can be controlled as small as necessary by choosing the design parameters. The attraction of the proposed formulation lies in its robustness to uncertainties and easiness to be implemented. Simulation results show the effectiveness of the proposed method.

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Appendix A: Proof of Lemma 1

Pre-multiplying both sides of equation (13) with

$$\operatorname{adj}\begin{bmatrix} a'(s) & -b'(s) \\ \mu(s) & \eta(s)b(s) \end{bmatrix}$$

yields

$$\Omega(s)\begin{bmatrix} y(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} \eta(s)b(s) \\ -\mu(s) \end{bmatrix} v(t) - \eta(s)b(s)\begin{bmatrix} b'(s) \\ a'(s) \end{bmatrix}\omega(t) + d(s)h(s)\begin{bmatrix} b'(s) \\ a'(s) \end{bmatrix} y_d(t), \quad (63)$$

where $\Omega(s)$ is a $2n$th order Hurwitz polynomial defined in (14). Thus, by using the relation of $y(t)$ in (63), $(\Delta a(s)/b(s))y(t)$ can be expressed as

$$\frac{\Delta a(s)}{b(s)} y(t) = \frac{\Delta a(s)\eta(s)}{\Omega(s)} v(t) - \frac{\Delta a(s)\eta(s)b'(s)}{\Omega(s)}\omega(t) + \frac{\Delta a(s)d(s)h(s)b'(s)}{\Omega(s)b(s)} y_d(t). \quad (64)$$

Since $\Delta a(s)\eta(s)/\Omega(s)$ and $\Delta a(s)\eta(s)b'(s)/\Omega(s)$ are strictly proper, it can be easily seen that $(\Delta a(s)/b(s))y(t)$ and its derivative $(s \cdot \Delta a(s)/b(s))y(t)$ are also bounded in the closed-loop system by observing the boundedness of $v(t)$, $\omega(t)$, $y_d(t)$ and the derivatives of $y_d(t)$.

As $u(t)$ is bounded in the closed-loop system, it can be seen that $(\Delta b(s)/b(s))u(t)$ and $(s\cdot\Delta b(s)/b(s))u(t)$ are also bounded in the closed-loop system by observing that $\Delta b(s)/b(s)$ is strictly proper.

Therefore, by the definition of $\bar{v}(t)$ in (7) and the assumptions, it can be concluded that the signal $\bar{v}(t)$ and its first-order derivative are bounded in the closed-loop system.

Appendix B: Proof of relation (26)

Choose the Lyapunov candidate $V_1(t)$ as

$$V_1(t)=\frac{1}{2}(\bar{y}(t))^2+\frac{|b_r|}{2\alpha_{r-1}}(\hat{C}_{r-1}(t)-C_{r-1})^2. \quad (65)$$

If $\bar{y}(t)>\sqrt{2\delta_1\hat{C}_{r-1}(t)/\lambda}$, then differentiating $V_1(t)$ yields

$$\begin{aligned}\dot{V}_1(t)&=-\lambda(\bar{y}(t))^2+\bar{y}(t)b_r\left\{\frac{1}{(s+\lambda)^{r-1}}\bar{v}(t)-w_1(t)\right\}\\&\quad+(\hat{C}_{r-1}(t)-C_{r-1})|b_r\bar{y}(t)|\\&=-\lambda(\bar{y}(t))^2+\left\{\bar{y}(t)\frac{b_r}{(s+\lambda)^{r-1}}\bar{v}(t)-|b_r\bar{y}(t)|C_{r-1}\right\}\\&\quad+\frac{\delta_1|b_r\bar{y}(t)|\hat{C}_{r-1}(t)}{|b_r\bar{y}(t)|+\delta}\\&\le-\lambda(\bar{y}(t))^2+\delta_1\hat{C}_{r-1}(t)\\&<-\delta_1\hat{C}_{r-1}(t).\end{aligned} \quad (66)$$

Thus, $V_1(t)$ decreases monotonically. Further, from (66), it can be seen that the condition $|\bar{y}(t)|\le\sqrt{2\delta_1\hat{C}_{r-1}(t)/\lambda}$ can be satisfied in finite time. Thus, there exists $t_1>0$ such that

$$|\bar{y}(t)|\le\sqrt{\frac{2\delta_1\hat{C}_{r-1}(t)}{\lambda}} \quad (67)$$

for $t>t_1$, and $V(t)$ (i.e. $|\bar{y}(t)|$ and $\hat{C}_{r-1}(t)$) is uniformly bounded for $0\le t\le t_1$. By (25) it can be seen that $\hat{C}_{r-1}(t)=\hat{C}_{r-1}(t_1)$ for $t>t_1$. Thus, it can be concluded that $\hat{C}_{r-1}(t)$ is uniformly bounded for all $t\ge 0$. Therefore, by (67), the conclusion (26) can be proved for $t>t_1$.

Appendix C: Proof of relation (27)

Differentiating the both sides of equation (23) yields

$$\ddot{\bar{y}}(t)+\lambda\dot{\bar{y}}(t)=b_r\psi_1(t), \quad (68)$$

where $\psi_1(t)$ is defined as

$$\psi_1(t)=\frac{s}{(s+\lambda)^{r-1}}\bar{v}(t)-\dot{\hat{C}}_{r-1}(t)\frac{b_r\bar{y}(t)}{|b_r\bar{y}(t)|+\delta_1}-b_r\hat{C}_{r-1}(t)\frac{b_r\dot{\bar{y}}(t)\delta_1}{(|b_r\bar{y}(t)|+\delta_1)^2}. \quad (69)$$

From Lemma 1, it can be seen that $\psi_1(t)$ is uniformly bounded. (The boundedness of the $\dot{\bar{v}}(t)$ is employed for the case $r=1$.) By the boundedness of $\dot{\bar{y}}(t)$ (see 23), it can be seen that $\ddot{\bar{y}}(t)$ is also bounded by observing (68).

Now, we divide the interval $[t_1,t]$ into the intervals $[\iota_i,\iota_{i+1}]$ $(i=1,\dots,p-1)$ such that $\dot{\bar{y}}(t)$ is non-negative or non-positive on the interval $[\iota_i,\iota_{i+1}]$, where $t_1=\iota_1<\iota_2<\dots<\iota_p=t$. By the boundedness of $\ddot{\bar{y}}(t)$, it can be seen that there exists a constant Γ such that $\min_i\{\iota_{i+1}-\iota_i\}\ge\Gamma$.

From (68), differentiating $(\dot{\bar{y}}(t))^2$ gives

$$\frac{d}{dt}(\dot{\bar{y}}(t))^2=-2\lambda(\dot{\bar{y}}(t))^2+2b_r\dot{\bar{y}}(t)\psi_1(t). \quad (70)$$

Thus, we have

$$\begin{aligned}(\dot{\bar{y}}(t))^2&=e^{-2\lambda(t-t_1)}(\dot{\bar{y}}(t_1))^2+2b_re^{-2\lambda t}\int_{t_1}^{t}e^{2\lambda\tau}\dot{\bar{y}}(\tau)\psi_1(\tau)\,d\tau\\&=e^{-2\lambda(t-t_1)}(\dot{\bar{y}}(t_1))^2\\&\quad+2b_re^{-2\lambda t}\sum_{i=1}^{p-1}\psi_1(\bar{\iota}_i)\int_{\iota_i}^{\iota_{i+1}}e^{2\lambda\tau}\dot{\bar{y}}(\tau)\,d\tau\\&=e^{-2\lambda(t-t_1)}(\dot{\bar{y}}(t_1))^2+2b_re^{-2\lambda t}\sum_{i=1}^{p-1}\psi_1(\bar{\iota}_i)\\&\quad\times\left(e^{2\lambda\iota_{i+1}}\bar{y}(\iota_{i+1})-e^{2\lambda\iota_i}\bar{y}(\iota_i)\right.\\&\quad\left.-2\lambda\int_{\iota_i}^{\iota_{i+1}}e^{2\lambda\tau}\bar{y}(\tau)\,d\tau\right),\end{aligned} \quad (71)$$

where the mean value theory is employed and $\iota_i\le\bar{\iota}_i\le\iota_{i+1}$.

Suppose $|\psi_1(t)|\le K$. Then, (71) gives

$$\begin{aligned}(\dot{\bar{y}}(t))^2&\le e^{-2\lambda(t-t_1)}(\dot{\bar{y}}(t_1))^2\\&\quad+2|b_r|e^{-2\lambda t}\sum_{i=1}^{p-1}|\psi_1(\bar{\iota}_i)(e^{2\lambda\iota_{i+1}}\bar{y}(\iota_{i+1})-e^{2\lambda\iota_i}\bar{y}(\iota_i))|\\&\quad+4|b_r|\lambda e^{-2\lambda t}\sum_{i=1}^{p-1}|\psi_1(\bar{\iota}_i)|\int_{\iota_i}^{\iota_{i+1}}e^{2\lambda\tau}|\bar{y}(\tau)|\,d\tau\\&\le e^{-2\lambda(t-t_1)}(\dot{\bar{y}}(t_1))^2+4K|b_r|e^{-2\lambda t}\gamma_{11}(\delta_1)\sum_{i=1}^{p-1}e^{2\lambda\iota_{i+1}}\end{aligned}$$

$$+ 4K|b_r|\lambda e^{-2\lambda t}\gamma_{11}(\delta_1)\sum_{i=1}^{p-1}\int_{\iota_i}^{\iota_{i+1}} e^{2\lambda\tau}\,\mathrm{d}\tau$$

$$= e^{-2\lambda(t-t_1)}(\dot{\bar{y}}(t_1))^2 + 4K|b_r|\gamma_{11}(\delta_1)\sum_{i=1}^{p-1} e^{-2\lambda(t-\iota_{i+1})}$$

$$+ 4K|b_r|\lambda e^{-2\lambda t}\gamma_{11}(\delta_1)\int_{t_1}^{t} e^{2\lambda\tau}\,\mathrm{d}\tau$$

$$\leq e^{-2\lambda(t-t_1)}(\dot{\bar{y}}(t_1))^2 + 4K|b_r|\gamma_{11}(\delta_1)\sum_{i=0}^{p-2} e^{-2\lambda i\Gamma}$$

$$+ 2K|b_r|\gamma_{11}(\delta_1)(1 - e^{-2\lambda(t-t_1)})$$

$$= e^{2\lambda(t-t_1)}(\dot{\bar{y}}(t_1))^2 + 4K|b_r|\gamma_{11}(\delta_1)\frac{1 - e^{-2\lambda\Gamma(p-1)}}{1 - e^{-2\lambda\Gamma}}$$

$$+ 2K|b_r|\gamma_{11}(\delta_1)(1 - e^{-2\lambda(t-t_1)}), \qquad (72)$$

where (26) is employed. Therefore, by observing (72), (27) can be proved.

Appendix D: Proof of relation (34)

From (28), it can be seen that relation (34) can be proved if we can prove that $w_1(t) - \hat{w}_1(t)$ is very small as t is sufficiently large. For this purpose, we consider the following trivial differential equation:

$$\dot{w}_1(t) + \lambda w_1(t) = (s + \lambda)w_1(t). \qquad (73)$$

By deriving the derivative of $w_1(t)$, it can be easily seen that $\hat{w}_1(t)$ is bounded by employing equation (23) and lemma 1. By considering the boundedness of $w_1(t)$, it can be concluded that $(s + \lambda)w_1(t)$ is also bounded. Assume that

$$\max_{t\geq t_0} |(s + \lambda)w_1(t)| = D_{r-2}, \qquad (74)$$

where D_{r-2} is an unknown constant.

Now, choose the Lyapunov candidate as

$$V_2(t) = \frac{1}{2}(w_1(t) - \hat{w}_1(t))^2 + \frac{1}{2\alpha_{r-2}}(\hat{C}_{r-2}(t) - D_{r-2})^2. \qquad (75)$$

If $|w_1(t) - \hat{w}_1(t)| > \sqrt{2\delta_2\hat{C}_{r-2}(t)/\lambda}$, then differentiating $V_2(t)$ yields

$$\dot{V}_2(t) = -\lambda(w_1(t) - \hat{w}_1(t))^2$$

$$+ (w_1(t) - \hat{w}_1(t))\{(s + \lambda)w_1(t) - w_2(t)\}$$

$$+ (\hat{C}_{r-2}(t) - D_{r-2})|w_1(t) - \hat{w}_1(t)|$$

$$= -\lambda(w_1(t) - \hat{w}_1(t))^2$$

$$+ ((w_1(t) - \hat{w}_1(t))\{(s + \lambda)w_1(t)\}$$

$$- D_{r-2}|w_1(t) - \hat{w}_1(t)|\}$$

$$+ \frac{\delta_2|w_1(t) - \hat{w}_1(t)|\hat{C}_{r-2}(t)}{|w_1(t) - \hat{w}_1(t) + \delta_2}$$

$$\leq -\lambda(w_1(t) - \hat{w}_1(t))^2 + \delta_2\hat{C}_{r-2}(t)$$

$$< -\delta_2\hat{C}_{r-2}(t). \qquad (76)$$

By referring the proof in appendix B, it can be similarly proved that $\hat{C}_{r-2}(t)$ is uniformly bounded and there exists $t_2 > 0$ such that

$$|w_1(t) - \hat{w}_1(t)| \leq \sqrt{\frac{2\delta_2\hat{C}_{r-2}(t)}{\lambda}}. \qquad (77)$$

Therefore, combining (28) and (77) yields the relation (34).

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