Pergamon

0005-1098(94)00119-7

Automatica, Vol. 31, No. 3, pp. 483–488, 1995
Elsevier Science Ltd
Printed in Great Britain
0005-1098/95 $9.50 + 0.00

Brief Paper

Combined Adaptive and Variable Structure Control for Constrained Robots*

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Key Words—Robot; constraints; variable structure control; adaptive control; efficiency.

Abstract—This paper addresses the problem of controller design for constrained robots. A control algorithm, described as a combination of the direct adaptive and the variable structure method, is presented for the trajectory tracking of an end-effector on a constrained surface with specified constraint forces. This scheme has certain advantages, both with respect to computational efficiency and with respect to design. With respect to computational efficiency, the calculation of the regressor is avoided. With respect to design, since the number of parameter update laws required in the adaptation is independent of the number of links of the robot, the difficulty of tuning numerous adaptation gains is avoided.

1. *Introduction*

To achieve a wider class of tasks that involve contact with a manipulator environment, issues of appropriate modeling and of effective new control strategies arise, since such contact usually results in the generation of external forces acting on the end-effector of a manipulator, and modifies the dynamic behavior of a manipulator. The mathematical model of a robot system in contact with its environment, when it is described by a holonomic smooth manifold, gives rise to a mathematical system composed of differential and algebraic equations (McClamroch and Huang, 1985). The control of such systems is called constrained robot control (see e.g. McClamroch and Wang, 1988; Mills and Goldenberg, 1989). The objective of control is to determine the input torques to achieve tracking for a desired trajectory on a constrained surface with specified constraint forces.

The initial study on controlling a constrained robot was done by Hemami and Wyman (1979). Later, a number of strong results were obtained, including those of Yoshikawa (1987), McClamroch and Wang (1988), Mills and Goldenberg (1989), Yoon and Salam (1988), Yun (1988), Young (1988) and Lozano and Brogliato (1990). A general theoretical framework of constrained motion control was proposed by McClamroch and Wang (1988), who developed a rigorous mathematical model for constrained robots explicitly incorporating the constraint description. This model was then used to develop a modified computed torque controller guaranteeing global asymptotic stability for position and force tracking. However, this modified computed torque controller required exact knowledge of robot dynamics. To deal with the uncertainty in the constrained robot model, adaptive controls (Carelli and Kelly, 1991; Jean and Fu, 1991; Han *et al.*, 1992) and variable structure control (Huang and Lin, 1992) were developed based on the model given by McClamroch and Wang (1988).

In Su *et al.* (1990, 1992), an alternative mathematical model for constrained robots was developed that embedded the constraint equation into the dynamic equation, resulting in an affine nonlinear system without constraints. Then adaptive control (Su *et al.*, 1990) and variable structure control (Su *et al.*, 1992) were proposed. Although these control laws can be shown to achieve asymptotic tracking, owing to the use of a regressor matrix, the computational complexity required for their implementation may be considerable. Therefore, a modified scheme (Jean and Fu 1991) to allow off-line computation of the regressors using the desired values instead of actual measurements was proposed. However, it should also be emphasized that use of the regressor is not the only method for controller design. In fact, robust controllers for robot motion control without the use of regressors are quite successful (see e.g. Stepanenko and Yuan 1992).

In this paper, based on the model of constrained robots established in Su *et al.*, (1990, 1992), a combined adaptive and variable structure control strategy is proposed. The justification for combining adaptive and variable structure methods was given by Narendra and Boskovic (1992), who demonstrated through a simple first-order linear plant that such a combination can overcome the principle drawbacks of a variable structure method, and is superior to any of the methods currently available in adaptive control. Therefore, a new class of robust adaptive control laws for nonlinear robotic systems has been developed. Compared with other robust methods (Carelli and Kelly, 1991; Su *et al.*, 1990, 1992; Jean and Fu, 1991; Han *et al.*, 1992), this scheme has certain advantages, both with respect to computational efficiency and with respect to design. With respect to computational efficiency, the calculation of the regressor is avoided. With respect to design, our algorithm requires the tuning of three updated parameters, independent of the number of links of the robot, whereas typical parameter adaptive algorithms can require updating of as many as 10 parameters for each link of the robot (Spong, 1993). Stability analysis shows that the controller guarantees the boundedness of the tracking error. Moreover, the tracking error can be made arbitrarily small.

2. *Constrained robot dynamics*

In the Euler–Lagrange formulation, the motion equation of an n-link rigid constrained robot can be expressed in joint space as

$$D(\mathbf{q})\ddot{\mathbf{q}} + B(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + G(\mathbf{q}) = \mathbf{u} + J^{T}(\mathbf{q})\lambda, \qquad (1)$$

$$\psi(\mathbf{q}) = 0, \qquad (2)$$

where $\mathbf{q} \in \mathbb{R}^n$ represents the generalized coordinates (joint positions), $\mathbf{u} \in \mathbb{R}^n$ is the vector of applied joint torques, $J^T(\mathbf{q}) \in \mathbb{R}^n$ is the vector of constraint forces in joint space, $\lambda \in \mathbb{R}^m$ is the associated Lagrange multiplier, $D(\mathbf{q}) \in \mathbb{R}^{n\times n}$ is the symmetric, bounded, positive-definite inertia matrix (Ghorbel *et al.*, 1993), the vector $B(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} \in \mathbb{R}^n$ represents

* Received 2 September 1992; revised 4 May 1993; revised 6 December 1993; received in final form 11 May 1994. This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor R. Lozano under the direction of Editor C. C. Hang. Corresponding author Dr Chun-Yi Su. Tel. 604 721 8662; Fax 604 721 6051; E-mail csu@sirius.uvic.ca.

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the centripetal and Coriolis torques, $G(\mathbf{q}) \in \mathbb{R}^n$ is the vector of gravitational torques, which is a bounded C^1 function, $\psi = 0$ is the constraint equation in joint space (the mapping $\psi: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is twice continuously differentiable), and $J(\mathbf{q}) = \partial\psi/\partial\mathbf{q}$ is the Jacobian matrix of the constraint equation (2). Two simplifying properties should be noted about this dynamic structure.

Property 1. A suitable definition of $B(\mathbf{q}, \dot{\mathbf{q}})$ makes the matrix $\dot{D} - 2B$ skew symmetric.

Property 2. There exist positive constants ρ_i, $i = 1, 2, 3$, such that

$$\|D(\mathbf{q})\| \le \rho_1,$$
$$\|B(\mathbf{q}, \dot{\mathbf{q}})\| \le \rho_2 \|\dot{\mathbf{q}}\|, \tag{3}$$
$$\|G(\mathbf{q})\| \le \rho_3.$$

Since $\psi(\mathbf{q}) = 0$ is identically satisfied, it is evident that $J\dot{\mathbf{q}} = 0$. Thus, the effect of the constraints on the end-effector can be viewed as restricting the robot dynamics to the manifold Ω defined by

$$\Omega = \{(\mathbf{q}, \dot{\mathbf{q}}) : \psi(\mathbf{q}) = 0; J(\mathbf{q})\dot{\mathbf{q}} = 0\}$$

rather than the space $\mathbb{R}^{2n}$.

Since the presence of m constraints causes the manipulator to lose m degrees of freedom, the manipulator is left with only $n - m$ degrees of freedom. In this case, $n - m$ linear independent coordinates are sufficient to characterize the constrained motion. Following Su *et al.* (1990, 1992), choosing $n - m$ out of n joint variables, denoted by

$$\mathbf{q}^1 = [q_1^1 \quad \cdots \quad q_{n-m}^1]^{\mathrm{T}}$$

to be the generalized coordinates, describes the constrained motion of the manipulator. The remaining joint variables are denoted by

$$\mathbf{q}^2 = [q_1^2 \quad \cdots \quad q_m^2]^{\mathrm{T}}.$$

By the implicit function theorem, the constraint equation (2) can always be expressed explicitly as (McClamroch and Wang, 1988)

$$\mathbf{q}^2 = \sigma(\mathbf{q}^1). \tag{4}$$

In the following development, we assume that this expression is global. We also assume that the elements of $\mathbf{q}^1$ are chosen to be the first $n - m$ components of $\mathbf{q}$. If this is not the case, (1) can always be reordered so that the first $n - m$ equations correspond to $\mathbf{q}^1$ and the last m to $\mathbf{q}^2$.

Still following Su *et al.* (1990, 1992), by defining

$$L(\mathbf{q}^1) = \begin{bmatrix} I_{n-m} \\ \dfrac{\partial\sigma(\mathbf{q}^1)}{\partial\mathbf{q}^1} \end{bmatrix}, \tag{5}$$

the dynamic model (1) of robots, when restricted to the constraint surface, can be expressed in reduced form as (Su *et al.*, 1990, 1992)

$$D(\mathbf{q}^1)L(q^1)\ddot{\mathbf{q}}^1 + B_1(\mathbf{q}^1, \dot{\mathbf{q}}^1)\dot{\mathbf{q}}^1 + G(\mathbf{q}^1) = \mathbf{u} + J^{\mathrm{T}}(\mathbf{q})\lambda, \tag{6}$$

where B_1 is defined as

$$B_1(\mathbf{q}^1, \dot{\mathbf{q}}^1) = D(\mathbf{q}^1)\dot{L}(\mathbf{q}^1) + B(\mathbf{q}^1, \dot{\mathbf{q}}^1)L(\mathbf{q}^1).$$

Three fundamental properties of the dynamic equation (6) have been established by Su *et al.* (1990, 1992) and proposed as follows.

Property 3. The matrix $A(\mathbf{q}^1) = L^{\mathrm{T}}(\mathbf{q}^1)D(\mathbf{q}^1)L(\mathbf{q})$ is symmetric and positive definite.

Property 4. The matrix $\dot{A}(\mathbf{q}^1) - 2L^{\mathrm{T}}(\mathbf{q}^1)B_1(\mathbf{q}^1, \dot{\mathbf{q}}^1)$ is skew symmetric.

Property 5.

$$J(\mathbf{q}^1)L(\mathbf{q}^1) = L^{\mathrm{T}}(\mathbf{q}^1)J^{\mathrm{T}}(\mathbf{q}^1) = 0.$$

The above properties are fundamental for designing the force/motion control law. Throughout this paper, the norm of a vector x is defined as

$$\|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

and that of a matrix A as the corresponding induced norm

$$\|A\| = \left(\max_{\text{eigenvalue}} A^{\mathrm{T}}A\right)^{1/2}.$$

3. *Reduced-order control*

3.1. *Robust adaptive controller design.* The considered adaptive controller design problem is stated as follows. Given a desired joint trajectory $\mathbf{q}_d$ and desired constraints force f_d, or, equivalently, a desired multiplier λ_d, that satisfy the imposed constraints, i.e. $\psi(\mathbf{q}_d) = 0$ and $f_d = J^{\mathrm{T}}(\mathbf{q}_d)\lambda_d$, determine a control law such that for all $(\mathbf{q}(0), \dot{\mathbf{q}}(0)) \in \Omega$, $\mathbf{q} \rightarrow \mathbf{q}_d$ and $f \rightarrow f_d$ as $t \rightarrow \infty$.

It should be noted that, since $\mathbf{q}^2 = \sigma(\mathbf{q}^1)$, it is only required to find a control law to satisfy $\mathbf{q}^1 \rightarrow \mathbf{q}_d^1$ as $t \rightarrow \infty$.

In order to derive the controller, the following assumptions are required.

Assumption A.1. The desired trajectory $\mathbf{q}_d(t)$ is chosen such that $\mathbf{q}_d$, $\dot{\mathbf{q}}_d$ and $\ddot{\mathbf{q}}_d$ are all bounded signals.

Assumption A.2. The minimum and maximum eigenvalues of the matrix $l^{\mathrm{T}}(\mathbf{q}^1)L(\mathbf{q}^1)$ satisfy

$$\lambda_{\min}(L^{\mathrm{T}}L) \ge \beta_1, \quad \lambda_{\max}(L^{\mathrm{T}}L) \le \beta_2(\mathbf{q}^1),$$

where β_1 is a strictly positive constant and $\beta_2(\mathbf{q}^1)$ is a known positive function, bounded for bounded $\mathbf{q}^1$.

Assumption A.3. There exists a known positive function $\beta_3(\mathbf{q}^1)$, bounded for bounded $\mathbf{q}^1$, such that

$$\left\|\frac{\partial L(q^1)}{\partial\mathbf{q}^1}\right\| \le \beta_3(\mathbf{q}^1) \quad \text{for} \quad \frac{\partial L(\mathbf{q}^1)}{\partial\mathbf{q}^1} \ne 0.$$

Remark. Assumptions A.2 and A.3 can always be satisfied, and need not be stated as assumptions, since $L(\mathbf{q}^1)$ is assumed to be known. One can always find, if it exists, a known constant β_1 and known functions $\beta_2(\mathbf{q}^1)$ and $\beta_3(\mathbf{q}^1)$ so that Assumptions A.2 and A.3 are satisfied. The purpose of using β_1, β_2 and β_3 is to try to make full use of the available knowledge so as to reduce the control gains.

With Property 2 and Assumptions A.2 and A.3, and noting that $\beta_2(\mathbf{q}^1) \ge 1$ owing to the structure property of L, the following property holds.

Property 6. For all $(\mathbf{q}^1, \dot{\mathbf{q}}^1) \in \mathbb{R}^{2(n-m)}$, the matrices $L^{\mathrm{T}}D(\mathbf{q}^1)L$, $L^{\mathrm{T}}B_1(\mathbf{q}^1, \dot{\mathbf{q}}^1)$ and $L^{\mathrm{T}}G(\mathbf{q}^1)$ satisfy

$$\|L^{\mathrm{T}}D(\mathbf{q}^1)L\| \le \rho_1\beta_2, \tag{7}$$
$$\|L^{\mathrm{T}}B_1(\mathbf{q}^1, \dot{\mathbf{q}}^1)\| \le \beta_2(\rho_2 + \rho_1\beta_3)\|\dot{\mathbf{q}}^1\|, \tag{8}$$
$$\|L^{\mathrm{T}}G(\mathbf{q}^1)\| \le \rho_3\beta_2, \tag{9}$$

where ρ_i, $i = 1, 2, 3$, are positive constants.

A sliding variable $\mathbf{s}_1 = [s_1^1 \quad \cdots \quad s_1^{n-m}] \in \mathbb{R}^{n-m}$ is defined as

$$\mathbf{s}_1 = \dot{\mathbf{e}}_m + \Lambda\mathbf{e}_m, \tag{10}$$

where $\mathbf{e}_m = \mathbf{q}^1 - \mathbf{q}_d^1$ denotes the tracking error and Λ is a positive-definite matrix whose eigenvalues are strictly in the right half complex plane.

The robust adaptive control law is then synthesized as

$$\mathbf{u} = -K_d L \mathbf{s}_1 - \frac{\beta_2}{\beta_1} L[\hat{\rho}_1 \|\ddot{\mathbf{q}}_r^1\| + \hat{\rho}_2(1+\beta_3)\|\dot{\mathbf{q}}^1\|\|\dot{\mathbf{q}}_r^1\| + \hat{\rho}_3] \times \frac{\mathbf{s}_1}{\|\mathbf{s}_1\| + \delta} - J^T(\mathbf{q}^1)\lambda_c, \tag{11}$$

where $K_d \in \mathbb{R}^{n\times n}$ is a positive-definite matrix, $\hat{\rho}_i$, $i = 1, 2, 3$, are the adaptive control gains, δ is a strictly positive constant and $\dot{\mathbf{q}}_r^1 \in \mathbb{R}^{n-m}$ is a vector of auxiliary signals defined by

$$\dot{\mathbf{q}}_r^1 = \dot{\mathbf{q}}_d^1 - \Lambda_d \mathbf{e}_m. \tag{12}$$

The force term λ_c in (11) is defined as

$$\lambda_c = \lambda_d - K_\lambda \mathbf{e}_\lambda, \tag{13}$$

where $K_\lambda \in \mathbb{R}^{m\times m}$ is a constant diagonal matrix of force control feedback gains, $\mathbf{e}_\lambda = \lambda - \lambda_d$.

The updated laws $\hat{\rho}_i$, $i = 1, 2, 3$, are defined as follows:

$$\dot{\hat{\rho}}_1 = \eta_1 \beta_2 m(\|\mathbf{x}\|) \frac{\|\mathbf{s}_1\|^2}{\|\mathbf{s}_1\| + \delta} \|\ddot{\mathbf{q}}_r^1\|, \tag{14}$$

$$\dot{\hat{\rho}}_2 = \eta_2 \beta_2 (1+\beta_3) m(\|\mathbf{x}\|) \frac{\|\mathbf{s}_1\|^2}{\|\mathbf{s}_1\| + \delta} \|\dot{\mathbf{q}}^1\| \|\dot{\mathbf{q}}_r^1\|, \tag{15}$$

$$\dot{\hat{\rho}}_3 = \eta_3 \beta_2 m(\|\mathbf{x}\|) \frac{\|\mathbf{s}_1\|^2}{\|\mathbf{s}_1\| + \delta}, \tag{16}$$

where $\eta_i > 0$, $i = 1, 2, 3$, are constants, determining the rates of adaptation, $m(\|\mathbf{x}\|)$ is a modulation function, which allows the controller transition between a nonadaptive and adaptive model and will be defined in the next section, and $\mathbf{x}^T = [\mathbf{e}_m^T \ \dot{\mathbf{e}}_m^T]$.

Remark. The controllers given by Carelli and Kelly (1991), Su *et al.* (1992) and Han *et al.* (1992) actually belong to the linear parameterization approach, where on-line calculation of regressors is required. For a typical six-joint robot this involves very intensive computations (Stepanenko and Yuan, 1992). However, first, the control law (11) gives an alternative control scheme without using regressors. Secondly, (11), with the help of the robot properties, is constructed with three update parameters independent of the number of links of the robots, avoiding the difficulty of tuning numerous adaptation gains for which there is presently no adequate theory (Spong, 1993).

3.2. *Stability analysis.* To carry out the stability analysis, we need to express the closed-loop system equation in terms of the sliding variable $\mathbf{s}_1$. Based on (10), using (6), and after some calculations, the following is obtained:

$$DL\dot{\mathbf{s}}_1 = \mathbf{u} - DL\ddot{\mathbf{q}}_r^1 - B_1\dot{\mathbf{q}}_r^1 - G - B_1\mathbf{s}_1 + J^T\lambda.$$

According to Property 4, this equation becomes

$$A\dot{\mathbf{s}}_1 = L^T DL\dot{\mathbf{s}}_1 = L^T\mathbf{u} - A\ddot{\mathbf{q}}_r^1 - L^T B_1 \dot{\mathbf{q}}_r^1 - L^T G - L^T B_1 \mathbf{s}_1. \tag{17}$$

3.2.1. *Known-parameter upper bounds.* If the upper bounds of the parameters of the system are known then $\hat{\rho}_i$, $i = 1, 2, 3$, in (11) can take their desired values, i.e. $\hat{\rho}_i = \rho_{di}$, $i = 1, 2, 3$, satisfying

$$\rho_{d1} \ge \left(1 + \frac{\delta}{\epsilon}\right)\rho_1,$$

$$\rho_{d2} \ge \left(1 + \frac{\delta}{\epsilon}\right)\max(\rho_1, \rho_2),$$

$$\rho_{d3} \ge \left(1 + \frac{\delta}{\epsilon}\right)\rho_3,$$

where ϵ is the admissible magnitude of $\|\mathbf{s}_1\|$ satisfying $\|\mathbf{s}_1\| \ge \epsilon$.

Lemma 1. Let $\epsilon > 0$ and choose $\hat{\rho}_i$ in (11) as $\hat{\rho}_i = \rho_{di}$, $i = 1, 2, 3$. Then the control (11) is continuous and the closed-loop system is uniformly ultimately bounded. Furthermore, the steady-state force $J^T\mathbf{e}_\lambda$ is inversely proportional to the norm of the matrix $K_\lambda + I$.

Proof. Consider the positive-definite function

$$V_1(t) = \tfrac{1}{2}\mathbf{s}_1^T A \mathbf{s}_1. \tag{18}$$

Differentiating (18) with respect to time along the trajectories of (17) and using Property 4 leads to

$$\begin{aligned} \dot{V}_1 &= \mathbf{s}_1^T A\dot{\mathbf{s}}_1 + \mathbf{s}_1^T L^T B_1 \mathbf{s}_1 \\ &= \mathbf{s}_1^T(L^T\mathbf{u} - A\ddot{\mathbf{q}}_r^1 - L^T B_1\dot{\mathbf{q}}_r^1 - L^T G) \\ &= -\mathbf{s}_1^T L^T K_d L \mathbf{s}_1 - \frac{\beta_2}{\beta_1}\mathbf{s}_1^T L^T L(\rho_{d1}\|\ddot{\mathbf{q}}_r^1\| + \rho_{d2}(1+\beta_3)\|\dot{\mathbf{q}}^1\| \\ &\quad \times \|\dot{\mathbf{q}}_r^1\| + \rho_{d3} + \rho_{d4}\|\mathbf{e}_m\|)\frac{\mathbf{s}_1}{\|\mathbf{s}_1\| + \delta} \\ &\quad + \mathbf{s}_1^T(-A\ddot{\mathbf{q}}_r^1 - L^T B_1 \dot{\mathbf{q}}_r^1 - L^T G). \end{aligned} \tag{19}$$

Using Assumptions A.2 and Property 6, (19) becomes

$$\begin{aligned} \dot{V}_1 &\le -\mathbf{s}_1^T L^T K_d L \mathbf{s}_1 - \beta_2[\rho_{d1}\|\ddot{\mathbf{q}}_r^1\| + \rho_{d2}(1+\beta_3) \\ &\quad \times \|\dot{\mathbf{q}}^1\|\|\dot{\mathbf{q}}_r^1\| + \rho_{d3}]\frac{\|\mathbf{s}_1\|^2}{\|\mathbf{s}_1\| + \delta} + \rho_1\beta_2\|\ddot{\mathbf{q}}_r^1\|\|\mathbf{s}_1\| \\ &\quad + \beta_2(\rho_2 + \rho_1\beta_3)\|\dot{\mathbf{q}}^1\|\|\dot{\mathbf{q}}_r^1\|\|\mathbf{s}_1\| + \rho_3\beta_2\|\mathbf{s}_1\| \\ &\le -\mathbf{s}_1^T L^T K_d L \mathbf{s}_1 \quad \text{for } \|\mathbf{s}_1\| \ge \epsilon. \end{aligned} \tag{20}$$

To complete the proof, it can be shown that there exist class-K functions $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ such that

$$\gamma_1(\|\mathbf{x}\|) \le V_1(\mathbf{x}, t) \le \gamma_2(\|\mathbf{x}\|), \tag{21}$$

where $\mathbf{x}^T = [\mathbf{e}_m^T \ \dot{\mathbf{e}}_m^T]$.

Since $L^T K_d L$ is symmetric and positive definite, Assumption A.2 implies that there exists a positive constant α so that $\alpha I \le L^T K_d L \ \forall \mathbf{q}^1 \in \mathbb{R}^{n-m}$. Then equation (20) shows that there also exists a class-K function $\gamma_3(\cdot)$ such that

$$\dot{V}_1(\mathbf{x}, t) \le -\gamma_3(\|\mathbf{x}\|) \tag{22}$$

is valid for $\mathbf{x}$ outside the ball $R(\epsilon) = \{\mathbf{x} : \|\mathbf{s}_1\| \le \epsilon\}$. Uniform ultimate boundedness of $\mathbf{x}$ thus follows using the results and terminology of Corless and Leitmann (1981). It should be mentioned that signals $\mathbf{q}^2$ are obtained through the relation $\mathbf{q}^2 = \sigma(\mathbf{q}^1)$.

Since $\mathbf{x}$ and therefore $\mathbf{e}_m$ and $\dot{\mathbf{e}}_m$ are bounded, use of the definitions of $\mathbf{e}_m$ and $\dot{\mathbf{q}}_r^1$ then yields that $\mathbf{q}_1$, $\dot{\mathbf{q}}_1$, $\dot{\mathbf{q}}_r^1$, and $\ddot{\mathbf{q}}_r^1$ are all bounded. Therefore, all signals on the right-hand side of (17) are bounded. On the basis of Assumption A.2 and the boundedness of D, A is thus bounded; hence, $\dot{\mathbf{s}}_1(t)$ is bounded. Using (10) and Assumption A.1 allows us to conclude that $\ddot{\mathbf{q}}_1$ is bounded. Substituting the control (11) into the reduced-order dynamic model (6) yields

$$\begin{aligned} J^T(\lambda - \lambda_c) &= \Big\{d(\mathbf{q}^1)L(\mathbf{q}^1)\ddot{\mathbf{q}}^1 + B_1(\mathbf{q}^1, \dot{\mathbf{q}}^1)\dot{\mathbf{q}}^1 + G(\mathbf{q}^1) + K_d L\mathbf{s}_1 \\ &\quad + \frac{\beta_2}{\beta_1}L[\rho_{d1}\|\ddot{\mathbf{q}}_r^1\| + \rho_{d2}(1+\beta_3)\|\dot{\mathbf{q}}^1\|\|\dot{\mathbf{q}}_r^1\| \\ &\quad + \rho_{d3} + \rho_{d4}\|\mathbf{e}_m\|]\ \frac{\mathbf{s}_1}{\|\mathbf{s}_1\| + \delta}\Big\} \\ &= \zeta(\mathbf{q}_1, \dot{\mathbf{q}}_1, \dot{\mathbf{q}}_r^1, \ddot{\mathbf{q}}_r^1). \end{aligned} \tag{23}$$

Therefore, ζ is a bounded function. Substituting λ_c into the above equation yields

$$J^T\mathbf{e}_\lambda = (K_\lambda + I)^{-1}\zeta. \tag{24}$$

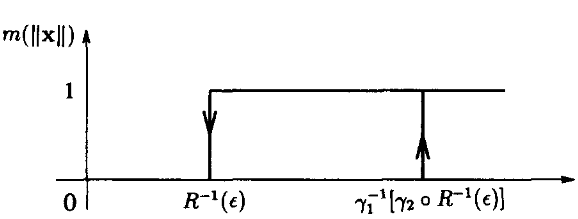

Fig. 1. Modulation $m(\|\mathbf{x}\|)$.

Therefore, the force tracking error $f - f_d$ is bounded and can be adjusted by changing the feedback gain K_λ. Thus, the lemma is proved. □

3.2.2. *Unknown-parameter upper bounds.* We now return to the main purpose of the paper, i.e. $\hat{\rho}_i$, $i = 1, 2, 3$, in (11) are unknown and adjusted by the adaptive law (14)–(16). Following the method given by Brogliato and Trofino-Neto (1992), let $m(\|\mathbf{x}\|)$ in (14)–(16) be chosen as depicted in Fig. 1. For the existence of solutions of the closed-loop systems, see Brogliato and Trofino-Neto (1992). Now, we can present the following stability theorem for the control law (11) and (14)–(16).

Theorem 1. Set $\epsilon(>0)$ as the admissible magnitude of $\mathbf{s}_1$. Then the control law (11) with the updated laws (14)–(16) in closed loop with the constrained robots modelled in the reduced form (6) guarantees that the following hold:
(i) the system state $\mathbf{x}$ is uniformly ultimately bounded, and the total time spent by $\mathbf{x}$ outside the ball $\{\mathbf{x}: \|\mathbf{x}\| \le \gamma_1^{-1}[\gamma_2 \circ R^{-1}(\epsilon)]\}$ is finite;
(ii) the steady-state force $J^T\mathbf{e}_\lambda$ is bounded and inversely proportional to the norm of the matrix $K_\lambda + I$.

Proof. Let us consider the positive-definition function

$$V(t) = V_1(t) + \tfrac{1}{2}\sum_{i=1}^{3}\frac{(\rho_{di} - \hat{\rho}_i)^2}{\eta_i}, \tag{25}$$

where $V_1(t)$ is defined in (18) and $\hat{\rho}_i$ are estimates of ρ_{di}.

In the case where $\mathbf{x}$ is outside $R(\epsilon)$, differentiating (25) with respect to time along the trajectories of (17) leads to

$$\begin{aligned}\dot{V} &= \dot{V}_1 + \sum_{i=1}^{3}\frac{(\rho_{di} - \hat{\rho}_i)(-\dot{\hat{\rho}}_i)}{\eta_i}\\ &\le -\gamma_3(\|\mathbf{x}\|) + (\rho_{d1} - \hat{\rho}_1)\beta_2\frac{\|\mathbf{s}_1\|^2}{\|\mathbf{s}_1\| + \delta}\|\ddot{\mathbf{q}}_r^1\|\\ &\quad + (\rho_{d2} - \hat{\rho}_2)\beta_2(1 + \beta_3)\frac{\|\mathbf{s}_1\|^2}{\|\mathbf{s}_1\| + \delta}\|\dot{\mathbf{q}}^1\|\,\|\dot{\mathbf{q}}_r^1\|\\ &\quad + (\rho_{d3} - \hat{\rho}_3)\beta_2\frac{\|\mathbf{s}_1\|^2}{\|\mathbf{s}_1\| + \delta} + \sum_{i=1}^{3}\frac{(\rho_i - \hat{\rho}_i)(-\dot{\hat{\rho}}_i)}{\eta_i}\\ &= -\gamma_3(\|\mathbf{x}\|) + [1 - m(\|\mathbf{x}\|)]\\ &\quad \times\Big[(\rho_{d1} - \hat{\rho}_1)\beta_2\frac{\|\mathbf{s}_1\|^2}{\|\mathbf{s}_1\| + \delta}\|\ddot{\mathbf{q}}_r^1\|\\ &\quad + (\rho_{d2} - \hat{\rho}_2)\beta_2(1 + \beta_3)\frac{\|\mathbf{s}_1\|^2}{\|\mathbf{s}_1\| + \delta}\|\dot{\mathbf{q}}^1\|\,\|\dot{\mathbf{q}}_r^1\|\\ &\quad + (\rho_{d3} - \hat{\rho}_3)\beta_2\frac{\|\mathbf{s}_1\|^2}{\|\mathbf{s}_1\| + \delta}\Big].\end{aligned} \tag{26}$$

Since $m(\|\mathbf{x}\|)$ are chosen as depicted in Fig. 1, following the results and argument given by Brogliato and Trofino-Neto (1992), the state $\mathbf{x}$ is uniformly ultimately bounded in the sense that given any $\epsilon > 0$, the total time spent by $\mathbf{x}$ outside the set $\gamma_1^{-1}[\gamma_2 \circ R^{-1}(\epsilon)]$ is finite, and $\hat{\rho}_i$, $i = 1, \ldots, 4$ remain bounded. Thus, (i) has been proved.

Since $\hat{\rho}_i$, $i = 1, 2, 3$, $\mathbf{x}$ and therefore $\mathbf{e}_m$ and $\dot{\mathbf{e}}_m$ remain bounded, the proof of (ii) is similar to that of Lemma 1, and is omitted here to save space. □

Remarks.
(1) For the controller design, the existence of ρ_{di}, $i = 1, 2, 3$, is necessary to guarantee the stability of the closed-loop system. However, these constants are not explicitly involved in the control inputs; the existence of ρ_{di} is sufficient for the validity of Theorem 1. The control inputs will rise to whatever level is necessary to ensure the stability of the overall system. The residual set can be made as small as desired by choosing the design parameters. The trade-off between the size of the residual set and the control gain can be determined by the choice of δ and ϵ.
(2) Since the choice of $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ is only related to the unknown inertia matrix L^TDL, one can always choose suitable $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$, although such a choice may be conservative. Moreover, even if $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ are unknown, following the results and argument of Brogliato and Trofino-Neto (1992), one can replace the value $\gamma_1^{-1}[\gamma_2 \circ R^{-1}(\varepsilon)]$ in the definition of $m(\|\mathbf{x}\|)$ by any constant $\xi \ge \gamma_1^{-1}[\gamma_2 \circ R^{-1}(\varepsilon)]$, and the above results will still hold. We should mention that, even with the very conservative choice of $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$, they are only related to the transition between the nonadaptive and adaptive modes, not to the controller gains. This differs from the case where the upper bounds are required to be known.
(3) Since the matrix L is known, β_1 and β_2 given in Assumption A.2 should be chosen in such a way as to reduce the magnitude of β_2/β_1. This, in turn, reduces the magnitude of the control law.
(4) No perfect force tracking objective is warranted. Nevertheless, part (ii) of Theorem 1 has a particular value, because improved steady-state constraint force accuracy is obtained with sufficiently high force gain. This result is similar to the results presented in Carelli and Kelly (1991).
(5) As $\delta \to 0$, the function $\mathbf{s}_1/(\|\mathbf{s}_1\| + \delta)$ eventually becomes discontinuous. In such a case, the controller becomes a typically variable structure control scheme, which may cause chattering phenomena. As a matter of fact, the control law (11) is just a smoothing realization of the switching function $\mathrm{sgn}(\|\mathbf{s}_1\|)$ to overcome chattering, which is undesirable in practice. It should be noted that if δ is chosen too small, such that the linear region of the function $\mathbf{s}_1/(\|\mathbf{s}_1\| + \delta)$ is very thin, there is a risk of exciting high frequency dynamics. This suggests that a trade-off must be made between the value of δ and trajectory-following requirements. An effective way to implement such a trade-off is to actually let δ vary depending on the state location. For a discussion of the tuning of δ, in this spirit, the reader is referred to Slotine (1984).

4. *Simulation results*

A two-link robotic manipulator with a circular path constraint, as given by Young (1988) and Su *et al.* (1990, 1992), is used to verify the validity of the control approach outlined in this paper. The original model, in the form of (1), can be written as

$$\begin{bmatrix} D_{11}(q_2) & D_{12}(q_2)\\ D_{12}(q_2) & D_{22}(q_2)\end{bmatrix}\begin{bmatrix}\ddot{q}_1\\ \ddot{q}_2\end{bmatrix} + \begin{bmatrix} -F_{12}(q_2)\dot{q}_2 & -F_{12}(q_2)(\dot{q}_1 + \dot{q}_2)\\ F_{12}(q_2)\dot{q}_1 & 0\end{bmatrix}\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix} + \begin{bmatrix} g_1(q_1, q_2)g\\ g_2(q_1, q_2)g\end{bmatrix} = \begin{bmatrix}u_1\\ u_2\end{bmatrix} + f, \tag{27}$$

where

$$D_{11}(q_2) = (m_1 + m_2)l_1^2 + m_2 l_2^2 + 2m_2 l_1 r_2 \cos q_2,$$
$$D_{12}(q_2) = m_2 l_2^2 + m_2 l_1 l_2 \cos q_2,$$
$$D_{22}(q_2) = m_2 l_2^2,$$
$$F_{12}(q_1) = m_2 l_1 l_2 \sin q_2,$$
$$g_1(q_1, q_2) = -(m_1 + m_2)l_1 \cos q_1 - m_2 l_2 \cos(q_1 + q_2),$$
$$g_2(q_1, q_2) = -m_2 l_2 \cos(q_1 + q_2).$$

The parameter values used are the same as those of Su *et al.* (1992), namely $l_1 = 1$, $l_2 = 0.8$, $m_1 = 0.5$ kg, $m_2 = 0.5$ kg, $J_1 = 5$ kg m and $J_2 = 5$ kg m.

The constraint is a circle in the work space (the (x, y) plane) whose center coincides with the axis of rotation of the first link. Figure 2 depicts the two-link manipulator and the constraint. The constraint, when expressed in terms of joint space, is

$$\psi(q) = l_1^2 + l_2^2 + 2l_1 l_2 \cos q_2 - r^2 = 0, \tag{28}$$

which has an unique constant solution for q_2:

$$q_2 = \cos^{-1}\left[\frac{r^2 - (l_1^2 + l_2^2)}{2l_1 l_2}\right] = q_2^*. \tag{29}$$

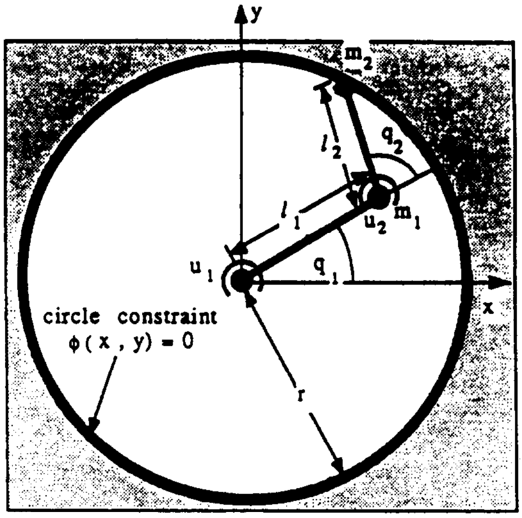

Fig. 2. Constrained robot system.

The Jacobian matrix of (28) is

$$J(q)=\begin{bmatrix}0\\ -2l_1l_2\sin q_2\end{bmatrix}. \qquad (30)$$

Therefore, the matrix L defined in (5) is

$$l(q^1)=[1 \quad 0]^{\mathrm{T}}. \qquad (31)$$

The constrained robot motion equation (6), when restricted to the circle, can be expressed as

$$\begin{bmatrix}D_{11}(q_2^*)\\ D_{22}(q_2^*)\end{bmatrix}\ddot{q}_1+\begin{bmatrix}0\\ F_{12}(q_2^*)\dot{q}_1\end{bmatrix}+\begin{bmatrix}g_1(q_1,q_2^*)g\\ g_2(q_1,q_2^*)g\end{bmatrix}=\begin{bmatrix}u_1\\ u_2\end{bmatrix}+\begin{bmatrix}0\\ -2l_1l_2\sin q_2^*\end{bmatrix}\lambda. \qquad (32)$$

The constraint forces are

$$f_1=0,\qquad f_2=-2l_1l_2\lambda\sin q_2^*. \qquad (33)$$

The control objective is to determine a feedback control so that the joint q_1 tracks the desired trajectory q_{1d} and maintains the constraint force f_2 to the desired f_d, where q_{1d} and f_d are assumed to be consistent with the imposed constraint.

Since $\lambda\to\lambda_d$ means $f_2\to f_d$, in this simulation q_{1d} and f_d are chosen as

$$q_{1d}=\begin{bmatrix}-90+52.5(1-\cos 1.26t)\\ 15\end{bmatrix},\qquad \lambda_d=10. \qquad (34)$$

The parameters β_1, β_2 and β_3 are chosen as $\beta_1, \beta_2=1$ and $\beta_3=0$. The control parameters are chosen as $K_d=5I$, $K_e=10I$, $\Lambda=4I$ and $K_\lambda=0.8I$. The adaptive gains are chosen as $\eta_1=4.5$, $\eta_2=4.5$ and $\eta_3=4.5$. δ and ϵ are chosen as $\delta=0.2$ and $\epsilon=0.2$.

Since trajectory tracking on the constrained surface with specified constraint force is of interest, the initial position and velocity of the manipulator are chosen on the desired trajectory:

$$q_1(0)=-90°,\quad q_2(0)=80°,\quad \dot{q}_1(0)=0,\quad \dot{q}_2(0)=0.$$

The initial constraint force is assumed as $f_2=0$, i.e. $\lambda=0$. The initial control gains $\hat{\rho}_i(0)$, $i=1,2,3$, are arbitrarily taken as $\hat{\rho}_1(0)=2$, $\hat{\rho}_2(0)=1$ and $\hat{\rho}_2(0)=1$.

The simulation is conducted for two cases: the fixed-parameters controller (dashed curves) and the adaptive controller (solid curves), where the fixed gains are chosen as $\rho_{1d}=3.5$, $\rho_{2d}=3.5$ and $\rho_{3d}=20$. Figure 3 shows the desired joint trajectory and Fig. 4 shows the contact force λ. The final maximum error with λ_d is 1.5 N. Figures 6 and 7 show the torques exerted at manipulator joints, and the changes in the adapted parameters are shown in Fig. 8. These results show that the control objective is achieved.

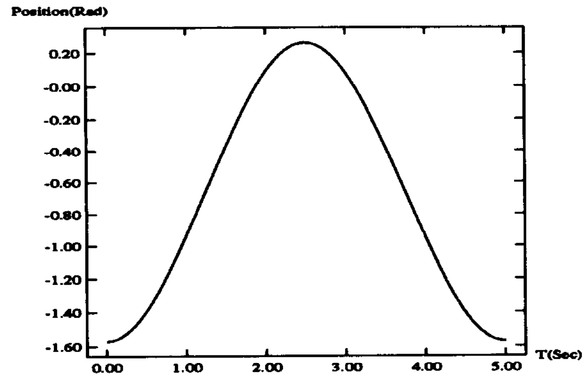

Fig. 3. Desired trajectory.

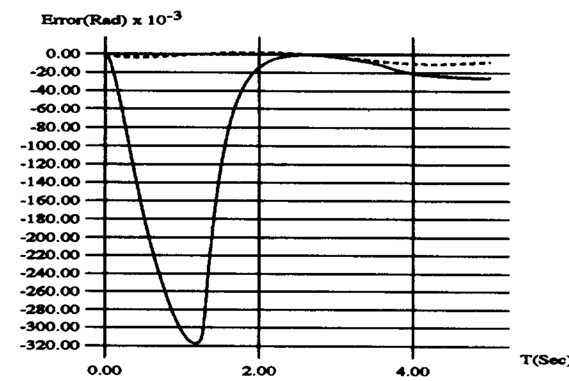

Fig. 4. Tracking errors.

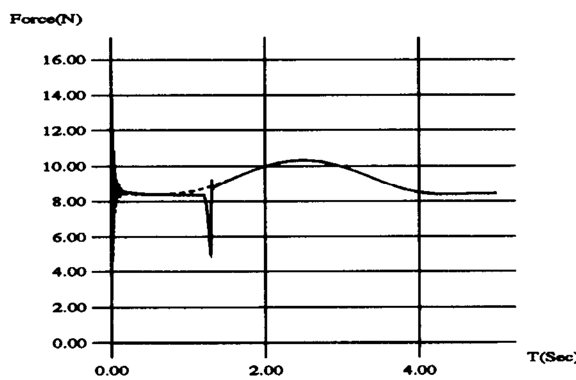

Fig. 5. Actual contact forces.

5. *Conclusions*

In this paper, we have derived an adaptive scheme, described as a combination of adaptive and variable structure control techniques, for constrained robots. The developed scheme actually utilizes the tracking error feedback with adaptive gains that grow until they reach whatever levels are necessary to cancel the nonlinearities. Compared with previous designs (parameter adaptive algorithms), clearly, the parameter adaptive algorithms are model-based methods, use more knowledge about the dynamic system and would be expected to work better. However, it is known that computation of the regressor is a time-consuming task and numerous parameters are required to be updated; they may

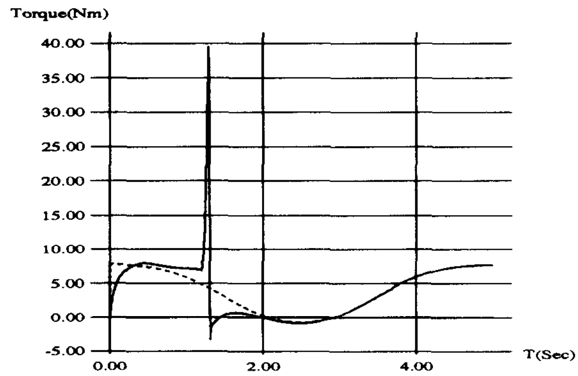

Fig. 6. Torque exerted at joint 1.

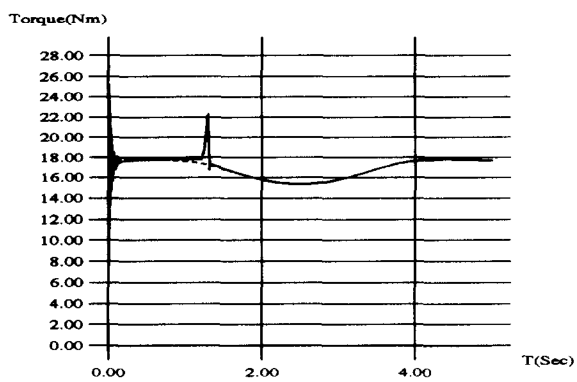

Fig. 7. Torque exerted at joint 2.

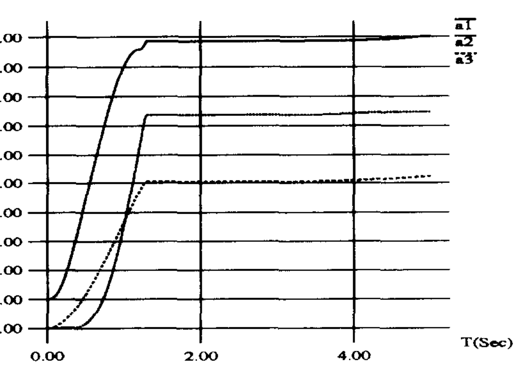

Fig. 8. Changes in parameters $\hat{\rho}_1$, $\hat{\rho}$ and $\hat{\rho}_3$.

also exhibit poor robustness to unmodeled dynamics and external disturbances unless the algorithms are modified (Spong, 1993). As an alternative to the parameter adaptive designs, our design has certain advantages, particularly with respect to computation, design and robustness. Therefore, the proposed scheme would be attractive in an environment where the computing capability has some limitations, knowledge of the mathematical model is somewhat lacking, and robustness to disturbances and unmodeled dynamics is of concern.

Acknowledgements—The authors wish to acknowledge the support of the Natural Science and Engineering Research Council of Canada, the Institute for Robotic and Intelligent Systems (IRIS) and Precarn Associates Inc. The authors also wish to thank anonymous referees for their helpful suggestions.

References

Brogliato, B. and A. Trofino-Neto (1992). Adaptive robust control of a class of nonlinear dynamic systems containing partially known uncertainties. In *Proc.* 1992 *American Control Conference*, Chicago, IL, pp. 2559–2563.

Carelli, R. and R. Kelly (1991). An adaptive impedance/force controller for robot manipulators. *IEEE Trans. Autom. Control*, **AC-36,** 967–971.

Corless, M. J. and G. Leitmann (1981). Continuous state feedback guaranteeing uniform ultimate boundedness for uncertain dynamic system. *IEEE Trans. Autom. Control*, **AC-26,** 1139–1144.

Ghorbel, F., B. Srinivasan and M. W. Spong (1993). On the positive definiteness and uniform boundedness of the inertia matrix of robot manipulators. In *Proc. 32nd IEEE Conf. on Decision and Control*, San Antonio, TX, pp. 1103–1108.

Hemami, H. and B. F. Wyman (1979). Modeling and control of constrained dynamic system with application to biped locomotion in the frontal plane. *IEEE Trans. Autom. Control*, **AC-24,** 526–535.

Han, Y., L. Lui, R. Lingarkar, N. A. Sinha and M. A. Elbestawi (1992). Adaptive control of constrained robotic manipulators. *Int. J. Robotics and Automation*, **7,** 50–56.

Huang, H. P. and M. Lin (1992). Robust force control for robotic manipulators. *Int. J. Control*, **56,** 631–653.

Jean, J. H., and L. C. Fu (1991). Efficient adaptive hybrid control strategies for robots in constrained manipulation. In *Proc.* 1991 *IEEE Int. Conf. on Robotics and Automation*, Sacramento, CA, pp. 1681–1686.

Lozano, R. and B. Brogliato (1990). Adaptive hybrid force position control for redundant manipulators. In *Proc. 29th IEEE Conf. on Decision and Control*, Honolulu, HI, pp. 1949–1950.

McClamroch, N. H. and H. P. Huang (1985). Dynamics of a closed chain manipulator. In *Proc.* 1985 *American Control Conference*, Boston, MA, pp. 50–55.

McClamroch, N. H. and D. Wang (1988). Feedback stabilization and tracking of constrained robots. *IEEE Trans. Autom. Control*, **AC-33,** 419–426.

Mills, J. K. and A. A. Goldenberg (1989). Force and position control of manipulators during constrained motion tasks. *IEEE Trans. Robotics and Automation*, **RA-5,** 30–46.

Narendra, K. S. and J. Boskovic (1992). A combined direct, indirect and variable structure method for robust adaptive control. *IEEE Trans. Autom. Control*, **AC-37,** 262–268.

Slotine, J.-J. E. (1984). Sliding controller design for nonlinear systems. *Int. J. Control*, **40,** 421–434.

Spong, M. W. (1993). Adaptive control of robot manipulators: design and robustness. In *Proc.* 1993 *American Control Conference*, San Francisco, CA, pp. 2826–2829.

Stepanenko, Y. and J. Yuan (1992). Robust adaptive control of a class of nonlinear mechanical systems with unbounded and fast-varying uncertainties. *Automatica*, **24,** 265–276.

Su, C. Y., T. P. Leung and Q. J. Zhou (1990). Adaptive control of robot manipulators under constrained motion. In *Proc. 29th IEEE Conf. on Decision and Control*, Honolulu, HI, pp. 2650–2655.

Su, C. Y., T. P. Leung and Q. J. Zhou (1992). Force/motion control of constrained robots using sliding mode. *IEEE Trans. Autom. Control*, **AC-37,** 668–672.

Yoon, C. S. and F. M. A. Salam (1989). Compliant control of constrained robot manipulators: stabilization on the constraint surface. In *Proc. 28th IEEE Conf. on Decision and Control*, Tampa, FL, pp. 1622–1627.

Yoshikawa, T. (1987). Dynamic hybrid position force control of robot manipulators: description of hand constraints and calculation of joint driving force. *IEEE Trans. Robotics and Automation*, **RA-3,** 386–392.

Young, K. D. (1988). Applications of sliding mode to constrained robot motion control. In *Proc.* 1988 *American Control Conference*, Atlanta, GA, pp. 912–917.

Yun, X. (1988). Dynamic state feedback control of constrained robot manipulator. In *Proc. 27th IEEE Conf. on Decision and Control*, Austin, TX, pp. 433–438.