A Nonlinear Disturbance Observer for Multivariable Systems and Its Application to Magnetic Bearing Systems

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***Abstract*—This paper proposes a new nonlinear-disturbance observer for multivariable minimum-phase systems with arbitrary relative degrees. The model uncertainties and the system nonlinearities are treated as disturbances. The estimation of individual disturbances is independent of each other and the derivatives of the disturbances can be independently estimated. The proposed formulation is inspired by the variable structure-control method and adaptive algorithms where the *a priori* information concerning the upper bounds of the disturbances and their derivatives is not required. The nonlinear-disturbance observer is robust to the types of disturbances. Stability analysis shows that the estimation error decreases exponentially to a steady value, which is determined by the design parameters. To illustrate the method, the proposed design is applied to a vertical-shaft magnetic-bearing system where the rotational disturbances and their derivatives are estimated based on a linearized model of the rotational motion. Simulation results show the effectiveness of the proposed method.**

***Index Terms*—Disturbance observer, magnetic bearing systems, minimum phase systems, multivariable systems.**

I. INTRODUCTION

RECENTLY, design of disturbance observers has received considerable attention and many different schemes have been suggested [1], [9], [10], [14], [16]. The motivation is suggested by the fact that if the disturbances can be estimated, then control of the uncertain dynamic systems with disturbances may become easier. For example, the controller with disturbance-cancellation functions can be easily constructed by using the estimated disturbances [1]. The construction of the disturbance observers, similar to that of state observers, has an important implication in practical applications.

Among the many suggested disturbance-observer techniques, the approximate differentiator type [9], [16] and H_∞-type [10] formulations have been popularly applied in the design of tracking controllers for motion-control systems. The procedure of the first approach closes an inner loop around the controlled plant to reject disturbances and force the input–output characteristics of this inner loop to approximate a "nominal" plant model at low frequencies. Tuning of the loop is accomplished through adjustment of a low-pass filter. Since the plant approximates a nominal model at low frequencies, overall closed-loop dynamics are usually well known and feedforward techniques are often applied. The second approach makes the best use of the merits in H_∞ control [10]. But there are some shortcomings to these approaches. A fatal one is that a satisfactory control can hardly be obtained when the types of the disturbances are unknown and the model uncertainties exist [9], [10]. Another is that these formulations can only cope with some low-frequency disturbances. It should be mentioned that an important result of estimating the frequency of a signal is proposed recently in [7].

Because of the excellent robustness to uncertainties of the variable structure systems (VSS) sliding-mode method [17]–[19], it is found a lot of applications in state estimation [3], disturbance estimation [1], fault detection [4], etc. This paper proposes a nonlinear-disturbance observer for multivariable systems based on the VSS approach. As a matter of fact, the scheme developed in [1], which is based on the VSS "equivalent control" method belongs to this category, where the knowledge of the upper bounds of the disturbances is required. However, the "equivalent control" method is not strict (because, on the sliding surface $S(t) = 0$, it cannot be proved that the derivative of $S(t)$ is also zero) and the *a priori* knowledge about the upper bound may not be easily obtained in practice.

A common feature of the disturbance observers in [1], [9] and [16] is that they are designed only for single-disturbance single-output uncertain systems. Even though the method developed in [10] can cope with multivariable systems, it can only deal with a very limited class of disturbances. It is well known that most of the practical control systems are multivariable systems, such as, to name a few, robots and magnetic bearing systems. However, extensions of the schemes in [1], [9], and [16] to multivariable uncertain systems may not be easy or, at least, are not obvious.

The main features of the disturbance observer in this paper are that it is developed for multivariable systems, it is robust to the types of the disturbances and the *a priori* knowledge concerning the bounds of the disturbances are not needed. The proposed nonlinear-disturbance observer is inspired by the VSS control method and adaptive algorithms. By first estimating the disturbance through a higher order filter, the disturbance through a lower order filter is inductively estimated. Stability analysis shows that the estimation error decreases exponentially to a steady value, which is determined by the design parameters.

To show the applicability of our design, the proposed disturbance-estimation scheme is applied to a vertical-shaft magnetic-bearing system, where the rotational disturbances and their derivatives are estimated based on a linearized model of the rotational motion. In the case of the magnetic-bearing model, some disturbances, such as unknown angular disturbances, are unmatched uncertainties [13], [14]. The present robust control theory is generally not applicable to the systems with such disturbances. In order to account for the unmatched

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disturbances in the controller design it is a challenging task to construct a disturbance observer that can estimate unmatched disturbances. Focusing on the magnetic-bearing system, a sliding mode observer for disturbance estimation was proposed in [14] and a controller was then developed to eliminate the disturbances. However, in [14], estimation of the derivatives of the disturbances is subjected to the condition that the disturbances are periodic signals with known frequencies. This may not be the case in practice.

The application of the proposed disturbance observer shows that the unmatched disturbances and their derivatives in the magnetic-bearing system can be estimated without the periodic condition. The simulation results demonstrate the effectiveness of our new method. It should be mentioned that the estimations of the disturbances and their derivatives are the crux of the formulated magnetic-bearing system and the purpose of this paper is to reveal the essential features of the disturbance estimation. It is well known that as long as the disturbances are estimated, it is straightforward to combine this estimation into the controller design [1], [9], [16]. Therefore, to emphasize the main issue, the discussion of corresponding controller designs will not be pursued here.

The organization of this paper is as follows. Section II gives the problem formulation. In Section III, the formulation for the disturbance estimation is proposed for multivariable minimum phase systems. In Section IV, the proposed method is applied to the magnetic bearing system. Simulation results show the effectiveness of the proposed method. Section V provides conclusions.

II. Problem Statement

Consider the uncertain dynamical system described by

$$\begin{bmatrix} s^{n_1}+a_{11}(s) & \cdots & a_{1p}(s) \\ \vdots & \vdots & \vdots \\ a_{p1}(s) & \cdots & s^{n_p}+a_{pp}(s) \end{bmatrix} \begin{bmatrix} y_1(t) \\ \vdots \\ y_p(t) \end{bmatrix} = \begin{bmatrix} b_{11}(s) & \cdots & b_{1r}(s) \\ \vdots & \vdots & \vdots \\ b_{p1}(s) & \cdots & b_{pr}(s) \end{bmatrix} \begin{bmatrix} u_1(t) \\ \vdots \\ u_r(t) \end{bmatrix} + \begin{bmatrix} k_{11}(s) & \cdots & k_{1m}(s) \\ \vdots & \vdots & \vdots \\ k_{p1}(s) & \cdots & k_{pm}(s) \end{bmatrix} \begin{bmatrix} v_1(t) \\ \vdots \\ v_m(t) \end{bmatrix} \tag{1}$$

where $y_i(t)$ $(i=1,\ldots,p)$ and $u_i(t)$ $(i=1,\ldots,r)$ denote the outputs and inputs, respectively; $v_i(t)$ $(i=1,\ldots,m)$ are the unknown signals composed of the disturbances, the model uncertainties and the nonlinear parts of the system; p, r, m, and n_i $(i=1,\ldots,p)$ are known positive integers; $a_{ij}(s), b_{ij}(s)$ and $k_{ij}(s)$ are known at most (n_i-1)th-order polynomials; s denotes the differential operator.

For compactness, we denote

$$A(s) = \begin{bmatrix} s^{n_1}+a_{11}(s) & \cdots & a_{1p}(s) \\ \vdots & \vdots & \vdots \\ a_{p1}(s) & \cdots & s^{n_p}+a_{pp}(s) \end{bmatrix} \tag{2a}$$

$$B(s) = \begin{bmatrix} b_{11}(s) & \cdots & b_{1r}(s) \\ \vdots & \vdots & \vdots \\ b_{p1}(s) & \cdots & b_{pr}(s) \end{bmatrix} \tag{2b}$$

$$K(s) = \begin{bmatrix} k_{11}(s) & \cdots & k_{1m}(s) \\ \vdots & \vdots & \vdots \\ k_{p1}(s) & \cdots & k_{pm}(s) \end{bmatrix} \tag{2c}$$

$$y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_p(t) \end{bmatrix}, \quad u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_r(t) \end{bmatrix}$$

$$v(t) = \begin{bmatrix} v_1(t) \\ \vdots \\ v_m(t) \end{bmatrix} \tag{3}$$

and call the signals $v_i(t)$ $(i=1,\ldots,m)$ the disturbances of the system.

In this paper, the following assumptions are made.

A1) Without loss of generality, it is assumed that $m=p$.

Remark 1: This assumption is just for the sake of simplicity. The results developed in this paper can easily be extended to the general case $p \geq m$. It should be emphasized that the goal is to develop a disturbance observer in a simpler setting that reveals its essential features.

A2) The disturbances $v_i(t)$ are bounded signals. Furthermore, $v_i(t)$ are piecewise differentiable and their first-order derivatives (at nondifferentiable points, it is meant the right- and left-hand derivatives) are bounded.

Remark 2: It should be noted that the *a priori* information about the bounds of the disturbances is not required in this paper. These bounds will be updated by adaptive algorithms.

Let $k(s) = \det\{K(s)\}$ and $q = \deg\{k(s)\}$. Suppose

$$k(s) = k_0 s^q + k_1 s^{q-1} \cdots + k_q \tag{4}$$

where $k_0 \neq 0$. It can be concluded that

$$q \leq n_1 - 1 + n_2 - 1 + \cdots + n_p - 1 = \sum_{i=1}^{p} n_i - p.$$

Further, the following assumption is made regarding $k(s)$.

A3) $k(s)$ is a Hurwitz polynomial.

Remark 3: Assumption A3) means that the system (1) is in minimum phase (with respect to the relation between the output and the disturbance). This can be verified by firstly rewriting system (1) in the observer canonical form

$$\begin{cases} \dot{X}_1 = A_1 X_1 + B_1 u + K_1 v \\ y = C_1 X_1 \end{cases}$$

and then checking the full rankness of

$$\begin{bmatrix} A_1 - \mathrm{Z}I & K_1 \\ C_1 & 0 \end{bmatrix}$$

for all complex numbers Z satisfying $\mathrm{Re(Z)} \geq 0$ [8].

The aim of this research is to estimate the disturbances $v_i(t)$ under the condition that they are bounded and their upper bounds are not available.

III. Formulation of the Disturbance Observer

Since $\det(\mathrm{adj}(K(s))) = (k(s))^{p-1}$ is also a Hurwitz polynomial, multiplying both sides of (1) with $\mathrm{adj}(K(s))$ yields

$$\{\mathrm{adj}\{K(s)\}\}A(s)y(t) = \{\mathrm{adj}\{K(s)\}\}B(s)u(t) + k(s) \times [v_1(t) \quad \cdots \quad v_p(t)]^T. \quad (5a)$$

Now, rewrite (5a) as

$$\begin{cases} s^{l_1}(\Xi_1 y(t)) = \Phi_1(s)y(t) + \Psi_1(s)u(t) + k(s)v_1(t) \\ \vdots \\ s^{l_p}(\Xi_p y(t)) = \Phi_p(s)y(t) + \Psi_p(s)u(t) + k(s)v_p(t) \end{cases} \quad (5b)$$

where, in the ith equation, Ξ_i is a row vector whose entries are constants, $\Phi_i(s)$ and $\Psi_i(s)$ are row vectors with appropriate dimensions whose entries are at most (l_i-1)th-order polynomials of s. Because $A(s), B(s)$, and $K(s)$ are known polynomials, $\Xi_i, \Phi_i(s), \Psi_i(s)$, and $k(s)$ can be calculated. Here, $l_i - q$ can be regarded as the "relative degree" (with respect to the relation between $\Xi_i y(t)$ and $v_i(t)$) of the ith equation in (5b).

For simplicity, let

$$l_i - q = \eta_i. \quad (6)$$

By observing (5b), it can be seen that the disturbances are separated. Now, we will estimate the disturbance $v_i(t)$ based on the ith equation in (5b). For this purpose, introduce an l_ith-order Hurwitz polynomial

$$f_i(s) = \frac{1}{k_0}k(s)(s+\lambda)^{\eta_i} \quad (7)$$

where λ is a positive constant.

Dividing both sides of the ith equation in (5b) with $f_i(s)$ yields

$$\Xi_i y(t) = \frac{f_i(s) - s^{l_i}}{f_i(s)}(\Xi_i y(t)) + \frac{\Phi_i(s)}{f_i(s)}y(t) + \frac{\Psi_i(s)}{f_i(s)}u(t) + \frac{k_0}{(s+\lambda)^{\eta_i}}v_i(t). \quad (8)$$

Multiplying both sides of (8) with $(s+\lambda)$ gives

$$\dot{z}_i(t) + \lambda z_i(t) = L_i(t) + \frac{k_0}{(s+\lambda)^{\eta_i - 1}}v_i(t) \quad (9)$$

where $z_i(t)$ and $L_i(t)$ are defined as

$$z_i(t) = \Xi_i y(t)$$
$$L_i(t) = (s+\lambda)\left\{\frac{f_i(s) - s^{l_i}}{f_i(s)}(\Xi_i y(t)) + \frac{\Phi_i(s)}{f_i(s)}y(t) + \frac{\Psi_i(s)}{f_i(s)}u(t)\right\}. \quad (10)$$

Remark 4: In the derivation of (8), the validity of canceling $k(s)$ is guaranteed by assumption A3). Further, $z_i(t)$ and $L_i(t)$ are computable signals since $z_i(t)$ is a linear expression of the outputs $y_i(t)(i = 1, \ldots, p)$ and $L_i(t)$ is composed of the filters of the inputs and outputs, where the fact that $(s+\lambda)(f_i(s) - s^{l_i})/(f_i(s))$ and the entries of $(s+\lambda)(\Phi_i(s))/(f_i(s))$ and $(s+\lambda)(\Psi_i(s))/(f_i(s))$ are all proper is employed.

Since $v_i(t)$ are bounded signals, it can be seen that signals $|(1/((s+\lambda)^j))v_i(t)|$ are also bounded for any positive integer j.

In the next theorem, firstly, $(1/((s+\lambda)^{\eta_i-1}))v_i(t)$ is estimated based on (9); secondly, $1/((s+\lambda)^{\eta_i-2})v_i(t)$ is inductively estimated by using the estimate of $(1/((s+\lambda)^{\eta_i-1}))v_i(t)$; and finally, the disturbance $v_i(t)$ is estimated.

Theorem 1: Construct the following differential equations:

$$\dot{\hat{z}}_i(t) + \lambda\hat{z}_i(t) = L_i(t) + k_0 w_{i,1}(t)$$
$$\hat{z}_i(t_0) = z_i(t_0) \quad (11)$$

$$\dot{\hat{w}}_{i,\mu_i-1}(t) + \lambda\hat{w}_{i,\mu_i-1}(t) = w_{i,\mu_i}(t)$$
$$\hat{w}_{i,\mu_i-1}(t_0) = 0 \quad (12)$$

where t_0 is the starting time; $\hat{z}_i(t)$ and $\hat{w}_{i,\mu_i-1}(t)$ $(1 < \mu_i \le \eta_i)$ are the variables which can be obtained by respectively solving (11) and (12); $w_{i,1}(t)$ and $w_{i,\mu_i}(t)(1 < \mu_i \le \eta_i)$ are the inputs described, respectively, by

$$w_{i,1}(t) = \hat{\omega}_{i,1}(t)\frac{k_0\{z_i(t) - \hat{z}_i(t)\}}{|k_0\{z_i(t) - \hat{z}_i(t)\}| + \delta_{i,1}} \quad (13)$$

and

$$w_{i,\mu_i}(t) = \hat{\omega}_{i,\mu_i}(t)\frac{w_{i,\mu_i-1}(t) - \hat{w}_{i,\mu_i-1}(t)}{|w_{i,\mu_i-1}(t) - \hat{w}_{i,\mu_i-1}(t)| + \delta_{i,\mu_i}} \quad (1 < \mu_i \le \eta_i) \quad (14)$$

$\delta_{i,j} > 0$ $(i = 1, \ldots, p; j = 1, \ldots, \eta_i)$ are design parameters which are usually chosen to be very small; $\hat{\omega}_{i,\mu_i}(t)$ $(1 \le \mu_i \le \eta_i)$ are updated by the following adaptive algorithms shown in (15) and (16) at the bottom of the page. $\hat{\omega}_{i,\mu_i}(t_0)$ can be chosen as any positive constants, α_{i,μ_i} are positive constants for $i = 1, \ldots, p, 1 \le \mu_i \le \eta_i$. It can be concluded that $w_{i,\mu_i}(t)$ are the corresponding approximate estimates of $(1/((s+\lambda)^{\eta_i-\mu_i}))v_i(t)$ for $1 \le \mu_i \le \eta_i$ as t is large enough, i.e., there exist $T_{i,\mu_i} \ge t_0$ and functions $\varepsilon_{i,\mu_i}(v_1, \ldots, v_{\mu_i}) > 0$ with the property $\lim_{\sum_{j=1}^{\mu_i}|v_j| \to 0}\varepsilon_{i,\mu_i}(v_1, \ldots, v_{\mu_i}) = 0$ such that

$$\left|\frac{1}{(s+\lambda)^{\eta_i-\mu_i}}v_i(t) - w_{i,\mu_i}(t)\right| < \varepsilon_{i,\mu_i}(\delta_{i,1}, \ldots, \delta_{i,\mu_i}) \quad (17)$$

for all $t \ge T_{i,\mu_i}$.

Proof: See Appendix A. □

Remark 5: For $i \ne j$, it can be seen that the estimation of $v_i(t)$ is independent of the estimation of $v_j(t)$. The formula-

$$\dot{\hat{\omega}}_{i,1}(t) = \begin{cases} 2\alpha_{i,1}|z_i(t) - \hat{z}_i(t)|, & \text{if } |k_0\{z_i(t) - \hat{z}_i(t)\}| > \delta_{i,1} \\ 0, & \text{otherwise} \end{cases} \quad (15)$$

$$\dot{\hat{\omega}}_{i,\mu_i}(t) = \begin{cases} 2\alpha_{i,\mu_i}|w_{i,\mu_i-1}(t) - \hat{w}_{i,\mu_i-1}(t)|, & \text{if } |w_{i,\mu_i-1}(t) - \hat{w}_{i,\mu_i-1}(t)| > \delta_{i,\mu_i}, \\ 0, & \text{otherwise} \end{cases} \quad (1 < \mu_i \le \eta_i). \quad (16)$$

tion of $w_{i,\mu_i}(t)$ is motivated by the VSS discontinuous control method [17].

Remark 6: The parameters $\delta_{i,j} > 0$ $(i = 1,\ldots,p; j = 1,\ldots,\eta_i)$ determine the estimating precision and the parameter $\lambda > 0$ determines the estimating speed. The parameters $\alpha_{i,\mu_i} > 0$ should be chosen large enough to adjust the estimated upper bounds $\hat{\omega}_{i,\mu_i}(t)$ rapidly for $i = 1,\ldots,p$ and $1 \leq \mu_i \leq \eta_i$. The estimation error for the disturbances can be designed to be arbitrarily small by choosing the design parameters. When the analog signals are implemented by a digital computer, the choice of the parameters $\delta_{i,j}$ is also limited by the sampling period [1].

IV. Application to Magnetic Bearing Systems

To show the applicability of our design, the proposed disturbance estimation scheme is applied to the rotational motion of a vertical shaft magnetic bearing system. The magnetic-bearing system is a device of electromagnets used to suspend a rotor without any contact. The technique of contactless support for rotors has become very important in a variety of industrial applications [2], [15].

Imbalance in the rotor mass will cause vibration in rotating machines. Balancing of the rotor is very difficult and there is often a residual imbalance. However, this imbalance problem can be solved by feedback control. One solution method is to compensate for the unbalance forces by generating electromagnetic canceling forces [5], [11], [12], [14]. The cancellation of the unbalance forces implies that these forces have to be estimated first. However, the estimation of unbalance forces is a nontrivial task and thus presents a challenge. In the following application, it will be shown that our method can contribute to a practical solution for the control of imbalance forces on magnetic bearings.

A. Model Description

The nonlinear model for the magnetic bearing assumes the rotor is a rigid floating body whose position is represented in the x, y, z coordinate frame about the rotor's center of mass. Let θ and ψ be the angular displacements of the rotor about the y-axis and the x-axis, respectively. The five degrees of freedom are represented accordingly as [15]

$$\ddot{x}_0 = \frac{1}{M}(f_{t2} - f_{t1} + f_{b2} - f_{b1} + f_{dx}) \tag{18a}$$

$$\ddot{y}_0 = \frac{1}{M}(f_{t3} - f_{t4} + f_{b3} - f_{b4} + f_{dy}) \tag{18b}$$

$$\ddot{z}_0 = \frac{1}{M}(-\beta z_0 - 2\varpi \dot{z}_0 + f_{b5} - f_{t5} + Mg + f_{dz}) \tag{18c}$$

$$\ddot{\theta} = \frac{-\rho J_a}{J_r}\dot{\psi} + \frac{\ell}{J_r}(f_{t1} - f_{t2} + f_{b2} - f_{b1} + f_{d\theta}) \tag{18d}$$

$$\ddot{\psi} = \frac{\rho J_a}{J_r}\dot{\theta} + \frac{\ell}{J_r}(f_{t3} - f_{t4} + f_{b4} - f_{b3} + f_{d\psi}) \tag{18e}$$

where M is the mass of the rotor; β is the axial stiffness coefficient; ϖ is the damping coefficient in the axial direction; g is the acceleration due to gravity; ρ is the rotor angular velocity; J_a and J_r are the moments of inertia in the axial and radial directions, respectively; ℓ is half of the longitudinal length. f_{dx}, f_{dy}, and f_{dz} are the translational disturbances, while $f_{d\theta}, f_{d\psi}$ are the rotational disturbances. The electromagnetic forces, f_j for $j = t1, t2, t3, t4, b1, b2, b3, b4$, produced by the jth electromagnet are expressed in terms of the air gap flux ϕ_j and the gap length g_j

$$f_j = \vartheta \phi_j^2 \left(1 + \frac{2g_j}{\pi h}\right) \tag{19}$$

where t and b represent top and bottom electromagnets, respectively; ϑ is a constant; and h is the pole width.

In this paper, we only consider the rotational disturbance identification for a vertical-shaft magnetic-bearing system. The translational disturbances can be independently estimated based on the dynamics in (18a)–(18c).

A linearization valid for rotational motion described in (18d) and (18e) is presented in [13]. This linear model has state variables $x_1 = \ell\theta, x_2 = \ell\psi, x_3 = \ell\dot{\theta}, x_4 = \ell\dot{\psi}, x_5 = \phi'_{t1} - \phi'_{b1}, x_6 = \phi'_{t3} - \phi'_{b3}$ where ϕ'_j represents the deviation of air gap flux from the nominal value at the jth electromagnet. The resulting is

$$\dot{X}_2 = A_2 X_2 + B_2(u + \nu) + E f_d \tag{20}$$

where the matrices A, B, and E are described by

$$A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{4c_2}{m_1} & 0 & 0 & -\rho\frac{J_a}{J_r} & \frac{2c_1}{m_1} & 0 \\ 0 & -\frac{4c_2}{m_1} & \rho\frac{J_a}{J_r} & 0 & 0 & \frac{2c_1}{m_1} \\ \frac{2d_2}{N} & 0 & 0 & 0 & -\frac{d_1}{N} & 0 \\ 0 & \frac{2d_2}{N} & 0 & 0 & 0 & -\frac{d_1}{N} \end{bmatrix} \tag{21a}$$

$$B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{1}{N} & 0 \\ 0 & \frac{1}{N} \end{bmatrix} \quad E = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_1} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{21b}$$

with $c_1 = 2\vartheta\phi_0(1 + (2G_0/\pi h)), c_2 = 2\vartheta\phi_0^2/\pi h, d_1 = 2RG_0/\Theta_0 SN, d_2 = 2R\phi_0/\Theta_0 SN$, and $m_1 = J_r/\ell^2, \phi_0$ is the nominal value for air gap flux, G_0 is the nominal value for gap length, R is the electromagnet coil resistance, Θ_0 is the permeability of free space, S is the area under one electromagnet pole, N is the number of turns in each electromagnet coil [13] and [14]. The state $X_2(t)$ is represented by $X_2(t) = [x_1\ x_2\ x_3\ x_4\ x_5\ x_6]^T$. The state $X_2(t)$ is unknown, however, its components x_1 and x_2 are measurable variables. The control input $u(t) = [u_1\ u_2]^T$ is the voltage potential applied between pairs of electromagnetic coils $(u_1 = e'_{t1} - e'_{b1}, u_2 = e'_{t3} - e'_{b3})$. In [13] and [14], it is assumed that the disturbance forces representing imbalance of the rotor about its geometrical axis can be represented as sinusoidal disturbances given by

$$f_d = [f_{d\theta} \quad f_{d\psi}]^T = \left[\frac{J_r - J_a}{\ell}\varphi\rho^2\cos(\rho t + \kappa) \quad \frac{J_r - J_a}{\ell}\varphi\rho^2\sin(\rho t + \kappa)\right]^T \tag{22}$$

where φ is the angular displacement between the geometrical and inertial axes (amount of dynamic imbalance), κ is the initial value of φ . $\nu(t) = [\nu_1\ \nu_2]^T$ represents the actuator noise. Since the actuator noise is matched, it will be neglected in the subsequent development knowing that the controller and observer can be made robust to matched disturbances [14], [17]–[19].

The disturbance force f_d representing imbalance of the rotor about its geometrical axis is an unmatched uncertainties. As pointed out in [14], the unmatched disturbance f_d and its derivative have to be estimated for the controller design. In the following, it will be shown that the disturbance f_d and its derivative can be estimated by using the result presented in Theorem 1.

B. Estimating the Disturbances and Their First-Order Derivatives

To begin, the system (20) is written in the input–output form

$$\left(\left(s^2+\frac{4c_2}{m_1}\right)\left(s+\frac{d_1}{N}\right)-\frac{4c_1d_2}{m_1N}\right)x_1 + \rho\frac{J_a}{J_r}s\left(s+\frac{d_1}{N}\right)x_2 = \frac{2c_1}{m_1N}u_1+\frac{1}{m_1}\left(s+\frac{d_1}{N}\right)f_{d\theta} \tag{23a}$$

$$\left(\left(s^2+\frac{4c_2}{m_1}\right)\left(s+\frac{d_1}{N}\right)-\frac{4c_1d_2}{m_1N}\right)x_2 - \rho\frac{J_a}{J_r}s\left(s+\frac{d_1}{N}\right)x_1 = \frac{2c_1}{m_1N}u_2+\frac{1}{m_1}\left(s+\frac{d_1}{N}\right)f_{d\psi}. \tag{23b}$$

Rewriting (23a) and (23b) in the form of (9) yields

$$\eta_1=\eta_2=2,\quad k_0=\frac{1}{m_1} \tag{24}$$

$$z_1=x_1,\quad v_1=f_{d\theta} \tag{25a}$$

$$L_1(t)=\frac{2\lambda s+\lambda^2-\frac{4c_2}{m_1}}{s+\lambda}x_1+\frac{4c_1d_2}{m_1N}\cdot\frac{1}{\left(s+\frac{d_1}{N}\right)(s+\lambda)}x_1 - \rho\frac{J_a}{J_r}\cdot\frac{s}{s+\lambda}x_2+\frac{2c_1}{m_1N}\cdot\frac{1}{\left(s+\frac{d_1}{N}\right)(s+\lambda)}u_1 \tag{25b}$$

$$z_2=x_2,\quad v_2=f_{d\psi} \tag{26a}$$

$$L_2(t)=\frac{2\lambda s+\lambda^2-\frac{4c_2}{m_1}}{s+\lambda}x_2+\frac{4c_1d_2}{m_1N}\cdot\frac{1}{\left(s+\frac{d_1}{N}\right)(s+\lambda)}x_2 + \rho\frac{J_a}{J_r}\cdot\frac{s}{s+\lambda}x_1+\frac{2c_1}{m_1N}\cdot\frac{1}{\left(s+\frac{d_1}{N}\right)(s+\lambda)}u_2. \tag{26b}$$

(25a) and (26a) also imply that $\Xi_1=[0,1]$ and $\Xi_2=[1,0]$.

From Theorem 1, for $i = 1, 2$, we construct the following observer equations:

$$\dot{\hat{z}}_i+\lambda\hat{z}_i=L_i(t)+\frac{1}{m_1}w_{i,1}(t)$$

$$\hat{z}_i(0)=x_i(0) \tag{27}$$

$$\dot{\hat{w}}_{i,1}(t)+\lambda\hat{w}_{i,1}(t)=w_{i,2}(t),\quad \hat{w}_{i,1}(0)=0 \tag{28}$$

where $w_{i,1}(t)$ and $w_{i,2}(t)$ are, respectively, determined by

$$w_{i,1}(t)=\hat{\omega}_{i,1}(t)\frac{\frac{1}{m_1}(x_i-\hat{z}_i)}{\left|\frac{1}{m_1}(x_i-\hat{z}_i)\right|+\delta_{i,1}} \tag{29}$$

and

$$w_{i,2}(t)=\hat{\omega}_{i,2}(t)\frac{w_{i,1}-\hat{w}_{i,1}}{|w_{i,1}-\hat{w}_{i,1}|+\delta_{i,2}} \tag{30}$$

with (31) and (32) shown at the bottom of the page.

Thus, $w_{1,2}(t)$ and $w_{2,2}(t)$ are, respectively, the estimates of $f_{d\theta}$ and $f_{d\psi}$.

In fact, Theorem 1 also gives a method of estimating the derivatives of a given signal (see Step 2 in the proof of Theorem 1). The derivatives $\dot{f}_{d\theta}$ and $\dot{f}_{d\psi}$ can be estimated by the following method.

For $i = 1, 2$, construct the observer equations

$$\dot{\hat{w}}_{i,2}(t)+\lambda\hat{w}_{i,2}(t)=w_{i,3}(t),\quad \hat{w}_{i,2}(0)=0 \tag{33}$$

where $w_{i,3}(t)$ are determined as

$$w_{i,3}(t)=\hat{\omega}_{i,3}(t)\frac{w_{i,2}-\hat{w}_{i,2}}{|w_{i,2}-\hat{w}_{i,2}|+\delta_{i,3}} \tag{34}$$

with (35) shown at the bottom of the page.

Thus, it can be concluded that $w_{1,3}(t)$ and $w_{2,3}(t)$ are the estimates of $(s+\lambda)f_{d\theta}$ and $(s+\lambda)f_{d\psi}$, respectively. Therefore, $\dot{f}_{d\theta}$ and $\dot{f}_{d\psi}$ can be estimated, respectively, by

$$\hat{\dot{f}}_{d\theta}=w_{1,3}(t)-\lambda w_{1,2}(t),\quad \hat{\dot{f}}_{d\psi}=w_{2,3}(t)-\lambda w_{2,2}(t). \tag{36}$$

Remark 7: The boundedness of $(s+\lambda)w_{i,2}(t)$ (for $i = 1, 2$) which can be derived from the definition of $w_{i,2}(t)$ is employed in the above formulation.

C. Simulation Results

The model parameter values of the vertical shaft magnetic bearing are shown in Table I. Since the input is just cancelled in the proposed formulation, the input is set to $u(t) = 0$. The parameters with appropriate units are chosen as $\lambda = 100, \delta_{1,1} = \delta_{2,1} = 1\times10^{-6}, \delta_{1,2} = \delta_{2,2} = 2\times10^{-5}, \delta_{1,3} = \delta_{2,3} = 8\times10^{-3}, \alpha_{i,j} = 1\times10^6$ $(i = 1, 2; j = 1, 2, 3)$; the initial values of

$$\dot{\hat{\omega}}_{i,1}(t)=\begin{cases}2\alpha_{i,1}|x_i-\hat{z}_i|, & \text{if } |x_i-\hat{z}_i|>m_1\delta_{i,1}\\ 0, & \text{otherwise}\end{cases} \tag{31}$$

$$\dot{\hat{\omega}}_{i,2}(t)=\begin{cases}2\alpha_{i,2}|w_{i,1}(t)-\hat{w}_{i,1}(t)|, & \text{if } |w_{i,1}(t)-\hat{w}_{i,1}(t)|>\delta_{i,2}\\ 0, & \text{otherwise.}\end{cases} \tag{32}$$

$$\dot{\hat{\omega}}_{i,3}(t)=\begin{cases}2\alpha_{i,3}|w_{i,2}(t)-\hat{w}_{i,2}(t)|, & \text{if } |w_{i,2}(t)-\hat{w}_{i,2}(t)|>\delta_{i,3}\\ 0, & \text{otherwise.}\end{cases} \tag{35}$$

TABLE I
MODEL PARAMETER VALUES

M	14.46 kg
β	1066 N/m
ϖ	.403 N s/m
ρ	$2\pi 120$ rad/s
J_a	.0136 kg m^2
J_r	.333 kg m^2
ℓ	.13 m
h	40 mm
ϑ	4.6755576*10^8
ϕ_0	2.09*10^{-4} Wb
G_0	.55 mm
R	14.7 ohms
Θ_0	$4\pi 10^{-7}$ H/m
S	1531.79 mm^2
N	400 turns
φ	10*10^{-6} rad
κ	0

$\hat{\omega}_{i,j}(t)$ are chosen as $\hat{\omega}_{i,j}(0) = 30$ for $i = 1, 2$ and $j = 1, 2, 3$. It should be noted that the parameters $\alpha_{i,j}$ $(i = 1, 2; j = 1, 2, 3)$ are chosen to be very large in order that the convergence speed of the estimating process may be very fast.

The computer simulation is carried out by using MATLAB, where the step size (sampling period) is chosen as 2×10^{-6} s. Fig. 1 shows the differences between the disturbances and their corresponding estimates. It can be seen that the convergence is very fast and the estimation errors are very small. Fig. 2 shows the differences between the derivatives of the disturbances and their corresponding estimates. It can be seen that very good estimates are obtained. These results confirm the validity of the proposed algorithm.

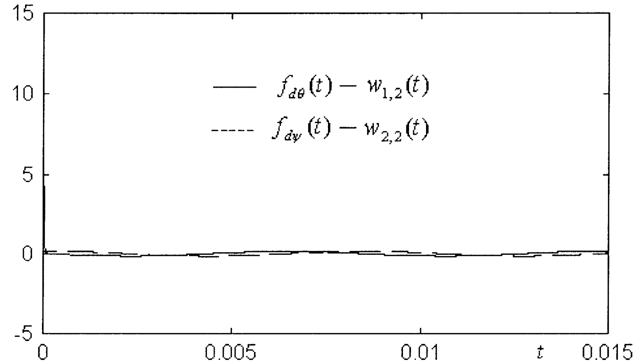

Fig. 1. Differences between the disturbances and their corresponding estimates.

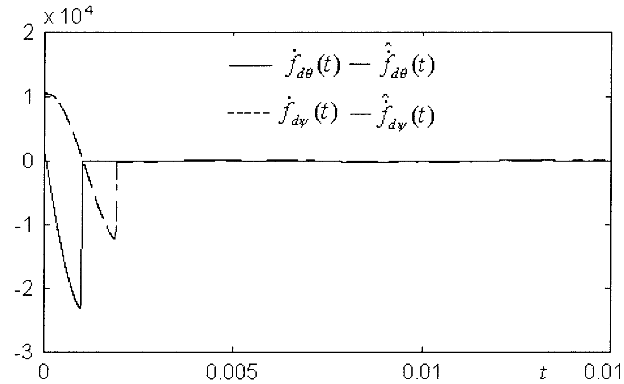

Fig. 2. Differences between the derivatives of the disturbances and their corresponding estimates.

D. Some Discussions

By using the estimates of the disturbances, the Luenberger-type state observer for the system (20) can be easily constructed as

$$\dot{\hat{X}}_2 = A_2\hat{X}_2 + B_2 u + E\begin{bmatrix} w_{1,2} \\ w_{2,2}\end{bmatrix} + PC_2(X_2(t) - \hat{X}_2(t)) \quad (37)$$

where $\hat{X}_2$ is the estimated state, C_2 is described by

$$C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (38)$$

$P \in R^{6\times 2}$ is chosen such that $(A_2 - PC_2)$ is a stable matrix. The roots of $(A_2 - PC_2)$ can be specified in advance.

Based on the estimates of the disturbances and their derivatives and the observed state, various control schemes can therefore be constructed, such as the sliding-mode controller [14], [17].

Instead of the above method, we can also express system (23a) and (23b) (equivalently system (20) with $\nu = 0$) in the observer canonical form with matched disturbances as

$$\dot{X}_3 = A_3X_3 + B_3\left(u + \frac{N}{2c_1}\left(\dot{f}_d + \frac{d_1}{N}f_d\right)\right) \quad (39)$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C_3X_3 \quad (40)$$

where $X_3(t)$ is the state variable in observer canonical form, matrices A_3, B_3, and C_3 are described, respectively, by

$$A_3 = \left[\begin{array}{ccc|ccc} -\frac{d_1}{N} & 1 & 0 & -\rho\frac{J_a}{J_r} & 0 & 0 \\ -\frac{4c_2}{m_1} & 0 & 1 & -\rho\frac{J_a d_1}{J_r N} & 0 & 0 \\ -\frac{4(c_2d_1 - c_1d_2)}{m_1 N} & 0 & 0 & 0 & 0 & 0 \\ \hline \rho\frac{J_a}{J_r} & 0 & 0 & -\frac{d_1}{N} & 1 & 0 \\ \rho\frac{J_a d_1}{J_r N} & 0 & 0 & -\frac{4c_2}{m_1} & 0 & 1 \\ 0 & 0 & 0 & -\frac{4(c_2d_1 - c_1d_2)}{m_1 N} & 0 & 0 \end{array}\right] \quad (41a)$$

$$B_3 = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \frac{2c_1}{m_1N} & 0 \\ \hline 0 & 0 \\ 0 & 0 \\ 0 & \frac{2c_1}{m_1N} \end{array}\right] \quad C_3 = \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ \hline 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{array}\right]^T. \quad (41b)$$

Based on (39), the Luenberger-type state observer can be similarly constructed by using the estimates of the disturbances and their derivatives. Further, the controller with the function of canceling the disturbances can then be synthesized [1], [9], [16], [17].

We should mention that no matter which method we use, it is obvious that the estimation of the disturbances and their derivatives is the crux of the formulated control system. The purpose of this paper is to reveal the essential features of the disturbance estimation. Therefore, to emphasize the main issue, the discussion of corresponding controller designs were not pursued here.

V. Conclusion

In this paper, a nonlinear disturbance observer is proposed for multivariable minimum phase systems with arbitrary relative degrees. The estimation of one disturbance is independent of the estimation of the others. Furthermore, the derivatives of the disturbances can also be independently estimated. The proposed disturbance observer is motivated by VSS method and adaptive algorithms. The *a priori* information about the upper and lower bounds of the disturbances and their derivatives is not required and these bounds are updated online by adaptive algorithms. The nonlinear disturbance observer is robust to the types of the disturbances. The estimation error decreases exponentially to a steady value, which is determined by the design parameters.

To show the effectiveness of the proposed method, the formulation is applied to a vertical-shaft magnetic-bearing system where the rotational disturbances and their derivatives are estimated based on a linearized model of the rotational motion. Simulation results confirm the validity of the proposed algorithm.

Appendix A
Proof of Theorem 1

Mathematical induction principle will be employed to prove this theorem.

1) Step 1: Based on (9), $(1/(s+\lambda)^{\eta_i-1})$ $(v_i(t)$ is estimated. For this purpose, let us consider the dynamical system described by (11). Combining (9) and (11) yields

$$\dot{\bar{z}}_i(t)+\lambda\bar{z}_i(t)=k_0\left\{\frac{1}{(s+\lambda)^{\eta_i-1}}v_i(t)-w_{i,1}(t)\right\} \quad (42)$$

where $\bar{z}_i(t)=z_i(t)-\hat{z}_i(t)$. It can be proved that $\bar{z}_i(t)$ and $\dot{\bar{z}}_i(t)$ are uniformly bounded, and there exist $T'_{i,1}\geq t_0, T_{i,1}\geq t_0$ (with $T_{i,1}\geq T'_{i,1}$) and a function $\gamma_{i,12}(v)>0$ with the property $\lim_{|v|\to 0}\gamma_{i,12}(v)=0$ such that

$$|\bar{z}_i(t)|\leq \delta_{i,1}/|k_0| \quad (\text{as } t\geq T'_{i,1}) \quad (43)$$
$$|\dot{\bar{z}}_i(t)|\leq \gamma_{i,12}(\delta_{i,1}) \quad (\text{as } t\geq T_{i,1}) \quad (44)$$

Relation (43) is proved in Appendix B. The proof of (44) is given in Appendix C.

Therefore, by combining (42)–(44), it can be concluded that there exists a function $\varepsilon_{i,1}(v)>0$ with the property $\lim_{|v|\to 0}\varepsilon_{i,1}(v)=0$ such that

$$\left|\frac{1}{(s+\lambda)^{\eta_i-1}}v_i(t)-w_{i,1}(t)\right|\leq\varepsilon_{i,1}(\delta_{i,1}) \quad (45)$$

as $t\geq T_{i,1}$.

2) Step 2: We will use $w_{i,1}(t)$ to estimate $(1/((s+\lambda)^{\eta_i-2}))v_i(t)$ by appealing to the next trivial equation

$$\frac{d}{dt}\left\{\frac{1}{(s+\lambda)^{\eta_i-1}}v_i(t)\right\}+\frac{\lambda}{(s+\lambda)^{\eta_i-1}}v_i(t) = \frac{1}{(s+\lambda)^{\eta_i-2}}v_i(t). \quad (46)$$

Corresponding to (46), the next differential equation is considered

$$\dot{\hat{w}}_{i,1}(t)+\lambda\hat{w}_{i,1}(t)=w_{i,2}(t), \quad \hat{w}_{i,1}(t_0)=0 \quad (47)$$

where $\hat{w}_{i,1}(t)$ is a signal which can be generated by solving the differential equation in (47), $w_{i,2}(t)$ is chosen as

$$w_{i,2}(t)=\hat{\omega}_{i,2}(t)\frac{w_{i,1}(t)-\hat{w}_{i,1}(t)}{|w_{i,1}(t)-\hat{w}_{i,1}(t)|+\delta_{i,2}} \quad (48)$$

where $\hat{\omega}_{i,2}(t)$ is defined in (16).

Denote $\bar{w}_{i,1}(t)=(1/(s+\lambda)^{\eta_i-1})v_i(t)-\hat{w}_{i,1}(t)$, then from (46) and (47), we have

$$\dot{\bar{w}}_{i,1}(t)+\lambda\bar{w}_{i,1}(t)=\frac{1}{(s+\lambda)^{\eta_i-2}}v_i(t)-w_{i,2}(t). \quad (49)$$

It can be proved that $\bar{w}_{i,1}(t)$ and $\dot{\bar{w}}_{i,1}(t)$ are uniformly bounded, and there exist $T'_{i,2}\geq t_0, T_{i,2}\geq t_0$ (with $T_{i,2}\geq T'_{i,2}$) and functions $\gamma_{i,2j}(v_1,v_2)>0$ $(j=1,2)$ with the property $\lim_{\sum_{j=1}^2|v_j|\to 0}\gamma_{i,2j}(v_1,v_2)=0$ such that

$$|\bar{w}_{i,1}(t)|\leq\gamma_{i,21}(\delta_{i,1},\delta_{i,2}) \quad (\text{as } t\geq T'_{i,2}) \quad (50)$$
$$|\dot{\bar{w}}_{i,1}(t)|\leq\gamma_{i,22}(\delta_{i,1},\delta_{i,2}) \quad (\text{as } t\geq T_{i,2}). \quad (51)$$

The proof of (50) is given in Appendix D. Relation (51) can be similarly proved by referring to the proof of (44) and employing the result in (50).

Therefore, combining (49)–(51) yields that there exists functions $\varepsilon_{i,2}(v_1,v_2)>0$ with the property $\lim_{\sum_{j=1}^2|v_j|\to 0}\varepsilon_{i,2}(v_1,v_2)=0$ such that

$$\left|\frac{1}{(s+\lambda)^{\eta_i-2}}v_i(t)-w_{i,2}(t)\right|\leq\varepsilon_{i,2}(\delta_{i,1},\delta_{i,2}) \quad (52)$$

as $t\geq T_{i,2}$.

3) Step ζ_i $(2<\zeta_i\leq\eta_i)$: Based on the trivial differential equation

$$\frac{d}{dt}\left\{\frac{1}{(s+\lambda)^{\eta_i-\zeta_i+1}}v_i(t)\right\}+\frac{\lambda}{(s+\lambda)^{\eta_i-\zeta_i+1}}v_i(t) = \frac{1}{(s+\lambda)^{\eta_i-\zeta_i}}v_i(t) \quad (53)$$

the following differential equation is considered:

$$\dot{\hat{w}}_{i,\zeta_i-1}(t)+\lambda\hat{w}_{i,\zeta_i-1}(t)=w_{i,\zeta_i}(t), \hat{w}_{i,\zeta_i-1}(t_0)=0 \quad (54)$$

where $\hat{w}_{i,\zeta_i-1}(t)$ is a signal which can be generated by solving the differential equation in (54), $w_{i,\zeta_i}(t)$ is chosen as

$$w_{i,\zeta_i}(t)=\hat{\omega}_{i,\zeta_i}(t)\frac{w_{i,\zeta_i-1}(t)-\hat{w}_{i,\zeta_i-1}(t)}{|w_{i,\zeta_i-1}(t)-\hat{w}_{i,\zeta_i-1}(t)|+\delta_{i,\zeta_i}} \quad (55)$$

where $\hat{\omega}_{i,\zeta_i}(t)$ is defined in (16).

By applying the results in the above (ζ_i-1) steps, we can similarly prove that there exists $T_{i,\zeta_i}>t_0$

and functions $\varepsilon_{i,\zeta_i}(v_1,\ldots,v_{\zeta_i}) > 0$ with the property $\lim_{\sum_{j=1}^{\zeta_i}|v_j|\to 0}\varepsilon_{i,\zeta_i}(v_1,\ldots,v_{\zeta_i}) = 0$ such that

$$\left|\frac{1}{(s+\lambda)^{\eta_i-\zeta_i}}v_i(t) - w_{i,\zeta_i}(t)\right| \le \varepsilon_{i,\zeta_i}(\delta_{i,1},\ldots,\delta_{i,\zeta_i}) \quad (56)$$

as $t \ge T_{i,\zeta_i}$.

By mathematical induction method, the theorem is proved (from Appendix C, it can be seen that the boundedness of $\dot{v}_i(t)$ is needed in the last step).

Appendix B
Proof of Relation (43)

Choose the Lyapunov candidate $V_{i,1}(t)$ as

$$V_{i,1}(t) = \frac{1}{2}(\bar{z}_i(t))^2 + \frac{|k_0|}{2\alpha_{i,1}}\left(\frac{1}{2}\hat{\omega}_{i,1}(t) - \omega_1\right)^2 \quad (57)$$

where the unknown constant ω_1 is the upper bound of $(1/(s+\lambda)^{\eta_i-1})v_i(t)$, i.e.,

$$\left|\frac{1}{(s+\lambda)^{\eta_i-1}}v_i(t)\right| \le \omega_1. \quad (58)$$

If $|k_0\bar{z}_i(t)| > \delta_{i,1}$, then differentiating $V_{i,1}(t)$ along (42) yields

$$\begin{aligned}\dot{V}_{i,1}(t) &= -\lambda(\bar{z}_i(t))^2 \\ &\quad + k_0\bar{z}_i(t)\left\{\frac{1}{(s+\lambda)^{\eta_i-1}}v_i(t) - w_{i,1}(t)\right\} \\ &\quad + (0.5\hat{\omega}_{i,1}(t) - \omega_1)|k_0\bar{z}_i| \\ &= -\lambda(\bar{z}_i(t))^2 \\ &\quad + \left\{k_0\bar{z}_i(t)\frac{1}{(s+\lambda)^{\eta_i-1}}v_i(t) - |k_0\bar{z}_i(t)|\omega_1\right\} \\ &\quad - \hat{\omega}_{i,1}(t)\frac{|k_0\bar{z}_i(t)|^2}{|k_0\bar{z}_i(t)| + \delta_{i,1}} + 0.5\hat{\omega}_{i,1}(t)|k_0\bar{z}_i(t)| \\ &= -\lambda(\bar{z}_i(t))^2 \\ &\quad + \left\{k_0\bar{z}_i(t)\frac{1}{(s+\lambda)^{\eta_i-1}}v_i(t) - |k_0\bar{z}_i(t)|\omega_1\right\} \\ &\quad - 0.5\hat{\omega}_{i,1}(t)|k_0\bar{z}_i(t)| + \hat{\omega}_{i,1}(t)\frac{|k_0\bar{z}_i(t)|\delta_{i,1}}{|k_0\bar{z}_i(t)| + \delta_{i,1}} \\ &\le -\lambda(\bar{z}_i(t))^2 + 0.5\hat{\omega}_{i,1}(t)|k_0\bar{z}_i(t)| \\ &\quad \times\left(\frac{2\delta_{i,1}}{|k_0\bar{z}_i(t)| + \delta_{i,1}} - 1\right) \\ &\le -\lambda(\bar{z}_i(t))^2 \\ &< -\lambda(\delta_{i,1}/k_0)^2 \end{aligned} \quad (59)$$

where the inequality (58) is employed. Equation (59) means that $V_{i,1}(t)$ decreases monotonically at a speed faster than $\lambda((\delta_{i,1})/(k_0))^2$ if the condition $|k_0\bar{z}_i(t)| > \delta_{i,1}$, i.e., $|\bar{z}_i(t)| > (\delta_{i,1})/(|k_0|)$ holds. Conversely, it can be concluded that the relation $|\bar{z}_i(t)| \le (\delta_{i,1})/(|k_0|)$ can be satisfied in finite time. Therefore, there exists $T'_{i,1} \ge t_0$ such that

$$|\bar{z}_i(t)| \le \frac{\delta_{i,1}}{|k_0|} \quad (60)$$

for $t \ge T'_{i,1}$, and $V_{i,1}(t)$ (thus, $|\bar{z}_i(t)|$ and $\hat{\omega}_{i,1}(t)$) is uniformly bounded for $t_0 \le t \le T'_{i,1}$. By the definition of $\hat{\omega}_{i,1}(t)$ and (60), it can be seen that $\hat{\omega}_{i,1}(t) = \hat{\omega}_{i,1}(T'_{i,1})$ for $t > T'_{i,1}$. Thus, it can be concluded that $\hat{\omega}_{i,1}(t)$ is uniformly bounded for all $t \ge t_0$. Therefore, relation (43) is proved.

Appendix C
Proof of Relation (44)

For $t \ge T'_{i,1}$, differentiating the both sides of (42) yields

$$\begin{aligned}\ddot{\bar{z}}_i(t) + \lambda\dot{\bar{z}}_i(t) &= k_0\frac{s}{(s+\lambda)^{\eta_i-1}}v_i(t) \\ &\quad - \hat{\omega}_{i,1}(t)\frac{(k_0)^2\dot{\bar{z}}_i(t)\delta_{i,1}}{(|k_0\bar{z}_i(t)| + \delta_{i,1})^2}\end{aligned} \quad (61)$$

where the fact $\dot{\hat{\omega}}_{i,1}(t) = 0$ is employed.

Thus, for $t \ge T'_{i,1}$, differentiating $(\dot{\bar{z}}_i(t))^2$ gives

$$\begin{aligned}&\frac{d}{dt}(\dot{\bar{z}}_i(t))^2 \\ &= -2\lambda(\dot{\bar{z}}_i(t))^2 + 2k_0\dot{\bar{z}}_i(t)\left(\frac{s}{(s+\lambda)^{\eta_i-1}}v_i(t)\right) \\ &\quad - 2\hat{\omega}_{i,1}(t)\frac{(k_0\dot{\bar{z}}_i(t))^2\delta_{i,1}}{(|k_0\bar{z}_i(t)| + \delta_{i,1})^2} \\ &\le -2\lambda(\dot{\bar{z}}_i(t))^2 + 2k_0\dot{\bar{z}}_i(t)\left(\frac{s}{(s+\lambda)^{\eta_i-1}}v_i(t)\right) \\ &\quad - \hat{\omega}_{i,1}(T'_{i,1})\frac{(k_0\dot{\bar{z}}_i(t))^2}{2\delta_{i,1}}\end{aligned} \quad (62)$$

where the facts $|k_0\bar{z}_i(t)| \le \delta_{i,1}$ and $\hat{\omega}_{i,1}(t) = \hat{\omega}_{i,1}(T'_{i,1})$ are employed.

By assumption (A2), it is easy to see that $(s/(s+\lambda)^{\eta_i-1})v_i(t)$ is uniformly bounded. Thus, there exists a constant $\mathrm{H} > 0$ such that

$$\left|\frac{s}{(s+\lambda)^{\eta_i-1}}v_i(t)\right| \le \mathrm{H}. \quad (63)$$

If $|\dot{\bar{z}}_i(t)| > (4\delta_{i,1}\mathrm{H})/(|k_0|\hat{\omega}_{i,1}(T'_{i,1}))$, then, from (62), it yields

$$\frac{d}{dt}(\dot{\bar{z}}_i(t))^2 \le -2\lambda(\dot{\bar{z}}_i(t))^2 < -32\lambda\left(\frac{\delta_{i,1}\mathrm{H}}{k_0\hat{\omega}_{i,1}(T'_{i,1})}\right)^2 \quad (64)$$

i.e., $|\dot{\bar{z}}_i(t)|$ decreases monotonically at a speed faster than the constant $32\lambda((\delta_{i,1}\mathrm{H})/(k_0\hat{\omega}_{i,1}(T'_{i,1})))^2$. Thus, the presupposition $|\dot{\bar{z}}_i(t)| > (4\delta_{i,1}\mathrm{H})/(|k_0|\hat{\omega}_{i,1}(T'_{i,1}))$ cannot be satisfied forever. Then, there exists an instant $T_{i,1} \ge T'_{i,1}$ such that

$$|\dot{\bar{z}}_i(t)| \le \frac{4\delta_{i,1}\mathrm{H}}{|k_0|\hat{\omega}_{i,1}(T'_{i,1})} \quad (65)$$

for all $t \ge T_{i,1}$. Therefore, relation (44) is proved.

$$|(s+\lambda)w_{i,1}(t)| = \left| \frac{\dot{\hat{\omega}}_{i,1}(t)\cdot k_0\bar{z}_i(t)}{|k_0\bar{z}_i(t)|+\delta_{i,1}} + \frac{\hat{\omega}_{i,1}(t)\cdot k_0\dot{\bar{z}}_i(t)\delta_{i,1}}{(|k_0\bar{z}_i(t)|+\delta_{i,1})^2} + \frac{\hat{\omega}_{i,1}(t)\cdot k_0\bar{z}_i(t)}{k_0\bar{z}_i(t)+\delta_{i,1}} \right| \le 4\mathrm{H} + \hat{\omega}_{i,1}(T_{i,1}) \triangleq D_{i,2} \tag{67}$$

APPENDIX D
PROOF OF RELATION (50)

From (45), it can be seen that relation (50) can be proved if we can prove that $w_{i,1}(t) - \hat{w}_{i,1}(t)$ is very small as t is sufficiently large. The following analysis is carried out for $t \ge T_{i,1}$.

Consider the following trivial differential equation:

$$\dot{w}_{i,1}(t) + \lambda w_{i,1}(t) = (s+\lambda)w_{i,1}(t). \tag{66}$$

First, let us show that $(s+\lambda)w_{i,1}(t)$ is uniformly bounded. By the definitions of $w_{i,1}(t)$, it gives (67) shown at the top of the page, where $\dot{\hat{\omega}}_{i,1}(t) = 0$ and relation (65) are employed, $D_{i,2}$ is an unknown constant.

Now, combining (47) and (66) yields

$$\frac{d}{dt}(w_{i,1}(t) - \hat{w}_{i,1}(t)) + \lambda(w_{i,1}(t) - \hat{w}_{i,1}(t)) = (s+\lambda)w_{i,1}(t) - w_{i,2}(t). \tag{68}$$

Choose the Lyapunov candidate as

$$V_{i,2}(t) = \frac{1}{2}(w_{i,1}(t) - \hat{w}_{i,1}(t))^2 + \frac{1}{2\alpha_{i,2}}\left(\frac{1}{2}\hat{\omega}_{i,2}(t) - D_{i,2}\right)^2. \tag{69}$$

If $|w_{i,1}(t) - \hat{w}_{i,1}(t)| > \delta_{i,2}$, then differentiating $V_{i,2}(t)$ yields

$$\begin{aligned}\dot{V}_{i,2}(t) &= -\lambda(w_{i,1}(t) - \hat{w}_{i,1}(t))^2 \\ &\quad + (w_{i,1}(t) - \hat{w}_{i,1}(t))\{(s+\lambda)w_{i,1}(t) - w_{i,2}(t)\} \\ &\quad + (0.5\hat{\omega}_{i,2}(t) - D_{i,2})|w_{i,1}(t) - \hat{w}_{i,1}(t)| \\ &= -\lambda(w_{i,1}(t) - \hat{w}_{i,1}(t))^2 \\ &\quad + (w_{i,1}(t) - \hat{w}_{i,1}(t))\{(s+\lambda)w_{i,1}(t)\} \\ &\quad - |w_{i,1}(t) - \hat{w}_{i,1}(t)|D_{i,2} \\ &\quad - 0.5\hat{\omega}_{i,2}(t)|w_{i,1}(t) - \hat{w}_{i,1}(t)| \\ &\quad + \hat{\omega}_{i,2}(t)|w_{i,1}(t) - \hat{w}_{i,1}(t)| \\ &\quad \cdot \frac{\delta_{i,2}}{|w_{i,1}(t) - \hat{w}_{i,1}(t)| + \delta_{i,2}} \\ &= -\lambda(w_{i,1}(t) - \hat{w}_{i,1}(t))^2 \\ &\quad + (w_{i,1}(t) - \hat{w}_{i,1}(t))\{(s+\lambda)w_{i,1}(t)\} \\ &\quad - |w_{i,1}(t) - \hat{w}_{i,1}(t)|D_{i,2} + \\ &\quad 0.5\hat{\omega}_{i,2}(t)|w_{i,1}(t) - \hat{w}_{i,1}(t)|\left(\frac{2\delta_{i,2}}{|w_{i,1}(t) - \hat{w}_{i,1}(t)| + \delta_{i,2}} - 1\right) \\ &\le -\lambda(w_{i,1}(t) - \hat{w}_{i,1}(t))^2 \\ &< -\lambda(\delta_{i,2})^2. \end{aligned} \tag{70}$$

By referring to the proof in Appendix B, it can be similarly proved that $|w_{i,1}(t) - \hat{w}_{i,1}(t)|$ and $\hat{\omega}_{i,2}(t)$ are uniformly bounded and there exists $T_{i,20} \ge t_0$, such that

$$|w_{i,1}(t) - \hat{w}_{i,1}(t)| \le \delta_{i,2} \tag{71}$$

for $t \ge T_{i,20}$. Therefore, combining (45) and (71) yields the relation (50).

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