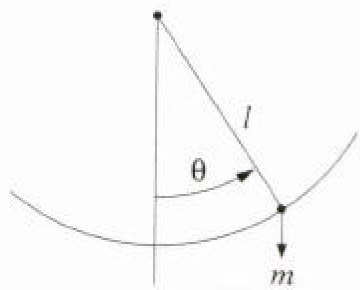


Figure 1.1: Pendulum.

Pendulum Equation

Consider the simple pendulum shown in Figure 1.1, where l denotes the length of the rod and m denotes the mass of the bob. Assume the rod is rigid and has zero mass. Let θ denote the angle subtended by the rod and the vertical axis through the pivot point. The pendulum is free to swing in the vertical plane. The bob of the pendulum moves in a circle of radius l. To write the equation of motion of the pendulum, let us identify the forces acting on the bob. There is a downward gravitational force equal to mg, where g is the acceleration due to gravity. There is also a frictional force resisting the motion, which we assume to be proportional to the speed of the bob with a coefficient of friction k. Using Newton's second law of motion, we can write the equation of motion in the tangential direction as

$$ml\ddot{\theta} = -mg\sin\theta - kl\dot{\theta}$$

Writing the equation of motion in the tangential direction has the advantage that the rod tension, which is in the normal direction, does not appear in the equation. Note that we could have arrived at the same equation by writing the moment equation about the pivot point. To obtain a state-space model of the pendulum, let us take the state variables as $x_1 = \theta$ and $x_2 = \dot{\theta}$. Then, the state equation is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 \end{aligned}$$

Example 3. Consider the pendulum equation without friction:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\left(\frac{g}{l}\right)\sin x_1\end{aligned}$$

and let us study the stability of the equilibrium point at the origin. A natural Lyapunov function candidate is the energy function

$$V(x) = \left(\frac{g}{l}\right)(1-\cos x_1) + \tfrac{1}{2}x_2^2$$

Clearly, $V(0) = 0$ and $V(x)$ is positive definite over the domain $-2\pi < x_1 < 2\pi$. The derivative of $V(x)$ along the trajectories of the system is given by

$$\begin{aligned}\dot{V}(x) &= \left(\frac{g}{l}\right)\dot{x}_1\sin x_1 + x_2\dot{x}_2\\ &= \left(\frac{g}{l}\right)x_2\sin x_1 - \left(\frac{g}{l}\right)x_2\sin x_1 = 0\end{aligned}$$

Thus, $V(x)$ satisfies conditions of Theorem and we conclude that the origin is stable. △

Example 3.2. Consider again the pendulum equation, but this time with friction:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\left(\frac{g}{l}\right)\sin x_1 - \left(\frac{k}{m}\right)x_2 \end{aligned}$$

Let us try again the energy as a Lyapunov function candidate.

$$\begin{aligned} V(x) &= \left(\frac{g}{l}\right)(1-\cos x_1) + \tfrac{1}{2}x_2^2 \\ \dot{V}(x) &= \left(\frac{g}{l}\right)\dot{x}_1 \sin x_1 + x_2\dot{x}_2 \\ &= -\left(\frac{k}{m}\right)x_2^2 \end{aligned}$$

$\dot{V}(x)$ is negative semidefinite. It is not negative definite because $\dot{V}(x) = 0$ for $x_2 = 0$ irrespective of the value of x_1; that is, $\dot{V}(x) = 0$ along the x_1-axis. Therefore, we can only conclude that the origin is stable. However, using the phase portrait of the pendulum equation, we have seen that when $k > 0$, the origin is asymptotically stable. The energy Lyapunov function fails to show this fact. Let us look for a Lyapunov function $V(x)$ that would have a negative definite $\dot{V}(x)$. Starting from the energy Lyapunov function, let us replace the term $\frac{1}{2}x_2^2$ by the more general quadratic form $\frac{1}{2}x^T P x$ for some 2×2 positive definite matrix P.

$$\begin{aligned} V(x) &= \tfrac{1}{2}x^T P x + \left(\frac{g}{l}\right)(1-\cos x_1) \\ &= \tfrac{1}{2}[x_1 \; x_2]\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \left(\frac{g}{l}\right)(1-\cos x_1) \end{aligned}$$

For the quadratic form $\frac{1}{2}x^T P x$ to be positive definite, the elements of the matrix P must satisfy

$$p_{11} > 0; \; p_{22} > 0; \; p_{11}p_{22} - p_{12}^2 > 0$$

The derivative $\dot{V}(x)$ is given by

$$\begin{aligned} \dot{V}(x) &= \left[p_{11}x_1 + p_{12}x_2 + \left(\frac{g}{l}\right)\sin x_1\right]x_2 \\ &\quad + (p_{12}x_1 + p_{22}x_2)\left[-\left(\frac{g}{l}\right)\sin x_1 - \left(\frac{k}{m}\right)x_2\right] \\ &= \left(\frac{g}{l}\right)(1-p_{22})x_2\sin x_1 - \left(\frac{g}{l}\right)p_{12}x_1\sin x_1 \\ &\quad + \left[p_{11} - p_{12}\left(\frac{k}{m}\right)\right]x_1x_2 + \left[p_{12} - p_{22}\left(\frac{k}{m}\right)\right]x_2^2 \end{aligned}$$

Now we want to choose p_{11}, p_{12}, and p_{22} such that $\dot{V}(x)$ is negative definite. Since the cross product terms $x_2 \sin x_1$ and $x_1 x_2$ are sign indefinite, we will cancel them by taking

$$p_{22} = 1; \;\; p_{11} = \left(\frac{k}{m}\right)p_{12}$$

With these choices, p_{12} must satisfy

$$0 < p_{12} < \left(\frac{k}{m}\right)$$

for $V(x)$ to be positive definite. Let us take $p_{12} = 0.5(k/m)$. Then, $\dot{V}(x)$ is given by

$$\dot{V}(x) = -\tfrac{1}{2}\left(\frac{g}{l}\right)\left(\frac{k}{m}\right)x_1 \sin x_1 - \tfrac{1}{2}\left(\frac{k}{m}\right)x_2^2$$

The term $x_1 \sin x_1 > 0$ for all $0 < |x_1| < \pi$. Defining a domain D by $D = \{x \in R^2 \mid |x_1| < \pi\}$, we see that $V(x)$ is positive definite and $\dot{V}(x)$ is negative definite over D. Thus, we conclude that the origin is asymptotically stable. △

In searching for a Lyapunov function in Example 3.2 we approached the problem in a backward manner. We investigated an expression for $\dot{V}(x)$ and went back to choose the parameters of $V(x)$ so as to make $\dot{V}(x)$ negative definite. This is a useful idea in searching for a Lyapunov function. A procedure that exploits this idea is known as the *variable gradient method.* To describe this procedure, let $V(x)$ be a scalar function of x and $g(x) = \nabla V = (\partial V/\partial x)^T$. The derivative $\dot{V}(x)$ along the trajectories is given by

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) = g^T(x) f(x)$$

The idea now is to try to choose $g(x)$ such that it would be the gradient of a positive definite function $V(x)$ and, at the same time, $\dot{V}(x)$ would be negative definite. It is not difficult to verify that $g(x)$ is the gradient of a scalar function if and only if the Jacobian matrix $[\partial g/\partial x]$ is symmetric, that is,

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}, \quad \forall\ i, j = 1, \ldots, n$$

Under this constraint, we start by choosing $g(x)$ such that $g^T(x) f(x)$ is negative definite. The function $V(x)$ is then computed from the integral

$$V(x) = \int_0^x g^T(y)\, dy = \int_0^x \sum_{i=1}^{n} g_i(y)\, dy_i$$

The integration is taken over any path joining the origin to x. Usually, this is done along the axes; that is,

$$\begin{aligned} V(x) \;=\; & \int_0^{x_1} g_1(y_1, 0, \ldots, 0)\, dy_1 + \int_0^{x_2} g_2(x_1, y_2, 0, \ldots, 0)\, dy_2 \\ & + \cdots + \int_0^{x_n} g_n(x_1, x_2, \ldots, x_{n-1}, y_n)\, dy_n \end{aligned}$$

By leaving some parameters of $g(x)$ undetermined, one would try to choose them to ensure that $V(x)$ is positive definite. The variable gradient method can be used to arrive at the Lyapunov function of Example 3.2. Instead of repeating the example, we illustrate the method on a slightly more general system.

Example 3.3. Consider the second-order system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -h(x_1) - a x_2 \end{aligned}$$

where $a > 0$, $h(\cdot)$ is locally Lipschitz, $h(0) = 0$ and $y h(y) > 0$ for all $y \neq 0$, $y \in (-b, c)$ for some positive constants b and c. The pendulum equation is a special case of this system. To apply the variable gradient method, we want to choose a second-order vector $g(x)$ that satisfies

$$\frac{\partial g_1}{\partial x_2} = \frac{\partial g_2}{\partial x_1}$$

$$\dot{V}(x) = g_1(x) x_2 - g_2(x)[h(x_1) + a x_2] < 0, \quad \text{for } x \neq 0$$

and

$$V(x) = \int_0^x g^T(y)\, dy > 0, \quad \text{for } x \neq 0$$

Let us try

$$g(x) = \begin{bmatrix} \alpha(x)x_1 + \beta(x)x_2 \\ \gamma(x)x_1 + \delta(x)x_2 \end{bmatrix}$$

where the scalar functions $\alpha(\cdot)$, $\beta(\cdot)$, $\gamma(\cdot)$, and $\delta(\cdot)$ are to be determined. To satisfy the symmetry requirement, we must have

$$\beta(x) + \frac{\partial \alpha}{\partial x_2}x_1 + \frac{\partial \beta}{\partial x_2}x_2 = \gamma(x) + \frac{\partial \gamma}{\partial x_1}x_1 + \frac{\partial \delta}{\partial x_1}x_2$$

The derivative $\dot{V}(x)$ is given by

$$\begin{aligned} \dot{V}(x) &= \alpha(x)x_1x_2 + \beta(x)x_2^2 - a\gamma(x)x_1x_2 \\ &\quad -a\delta(x)x_2^2 - \delta(x)x_2h(x_1) - \gamma(x)x_1h(x_1) \end{aligned}$$

To cancel the cross product terms, we choose

$$\alpha(x)x_1 - a\gamma(x)x_1 - \delta(x)h(x_1) = 0$$

so that

$$\dot{V}(x) = -[a\delta(x) - \beta(x)]x_2^2 - \gamma(x)x_1h(x_1)$$

To simplify our choices, let us take $\delta(x) = \delta =$ constant, $\gamma(x) = \gamma =$ constant, and $\beta(x) = \beta =$ constant. Then, $\alpha(x)$ depends only on x_1, and the symmetry requirement is satisfied by choosing $\beta = \gamma$. The expression for $g(x)$ reduces to

$$g(x) = \begin{bmatrix} a\gamma x_1 + \delta h(x_1) + \gamma x_2 \\ \gamma x_1 + \delta x_2 \end{bmatrix}$$

By integration, we obtain

$$\begin{aligned} V(x) &= \int_0^{x_1} [a\gamma y_1 + \delta h(y_1)]\, dy_1 + \int_0^{x_2} (\gamma x_1 + \delta y_2)\, dy_2 \\ &= \tfrac{1}{2}a\gamma x_1^2 + \delta \int_0^{x_1} h(y)\, dy + \gamma x_1 x_2 + \tfrac{1}{2}\delta x_2^2 \\ &= \tfrac{1}{2}x^T P x + \delta \int_0^{x_1} h(y)\, dy \end{aligned}$$

where

$$P = \begin{bmatrix} a\gamma & \gamma \\ \gamma & \delta \end{bmatrix}$$

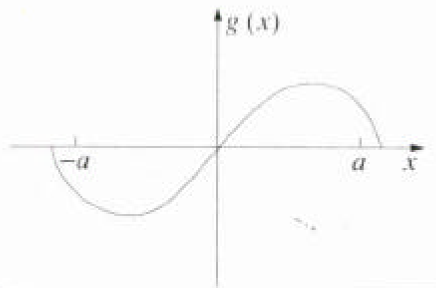
Choosing $\delta > 0$ and $0 < \gamma < a\delta$ ensures that $V(x)$ is positive definite and $\dot{V}(x)$ is negative definite. For example, taking $\gamma = ak\delta$ for $0 < k < 1$ yields the Lyapunov function

$$V(x) = \frac{\delta}{2}x^T \begin{bmatrix} ka^2 & ka \\ ka & 1 \end{bmatrix} x + \delta \int_0^{x_1} h(y)\, dy$$

which satisfies conditions of Theorem 2 over the domain $D = \{x \in R^2 \mid -b < x_1 < c\}$. Therefore, the origin is asymptotically stable. △

Consider the first-order differential equation

$$\dot{x} = -g(x)$$



where $g(x)$ is locally Lipschitz on $(-a, a)$ and satisfies

$$g(0) = 0; \quad xg(x) > 0, \quad \forall\ x \neq 0, \ x \in (-a, a)$$

The system has an isolated equilibrium point at the origin. It is not difficult in this simple example to see that the origin is asymptotically stable, because solutions starting on either side of the origin will have to move toward the origin due to the sign of the derivative $\dot{x}$. To arrive at the same conclusion using Lyapunov's theorem, consider the function

$$V(x) = \int_0^x g(y)\ dy$$

Over the domain $D = (-a, a)$, $V(x)$ is continuously differentiable, $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$. Thus, $V(x)$ is a valid Lyapunov function candidate. To see whether or not $V(x)$ is indeed a Lyapunov function, we calculate its derivative along the trajectories of the system.

$$\dot{V}(x) = \frac{\partial V}{\partial x}[-g(x)] = -g^2(x) < 0, \ \forall\ x \in D - \{0\}$$

Thus, by Theorem 3.1 we conclude that the origin is asymptotically stable.

△